# 1-D matrix method

We can expand the simple plane-wave scattering for 1-D examples that we've seen into a more versatile matrix approach that can be used to handle many interesting 1-D problems.

The basic idea is that we can break a problem having a complicated potential profile into a sequence of constant potential regions. Since we already know the TISE solutions for regions of constant potential, the problem boils down to connecting the solutions at each interface. A matrix approach lends itself well to this type of problem.
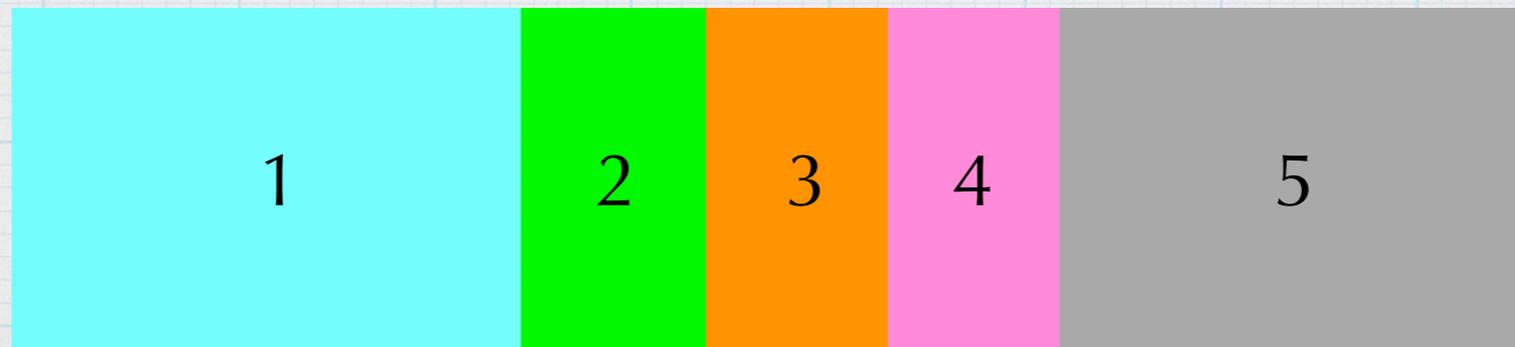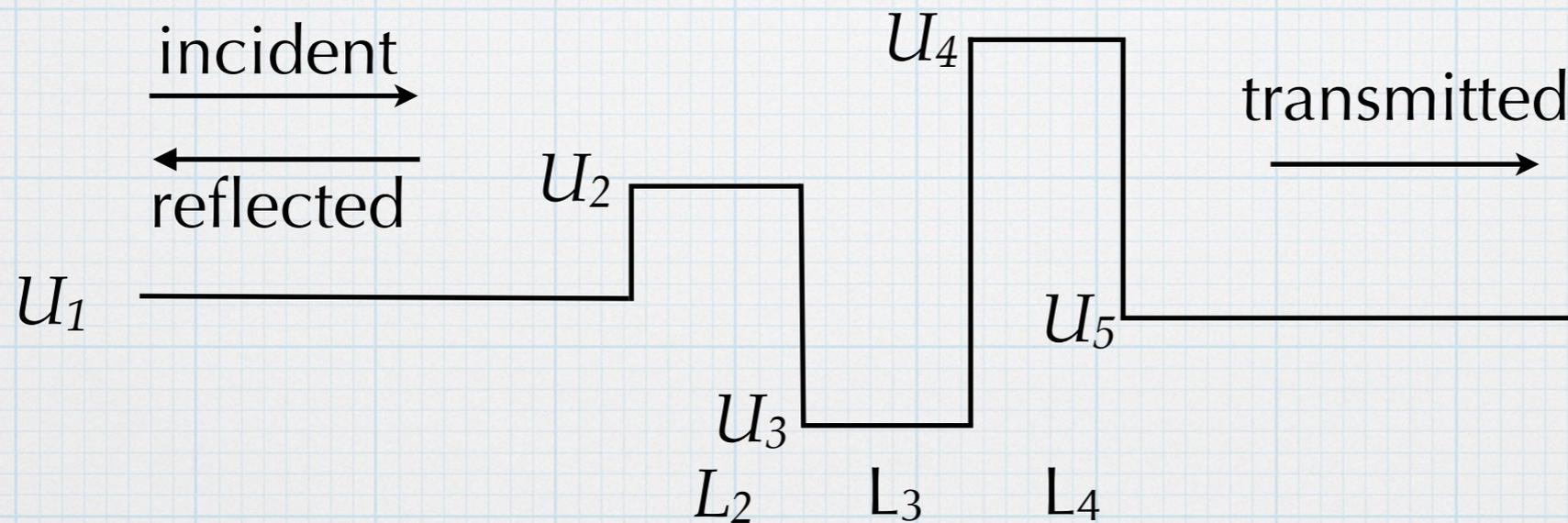
Also, we will see that the method can be used to find energy levels in confining quantum wells.

etc

E_2

E_1
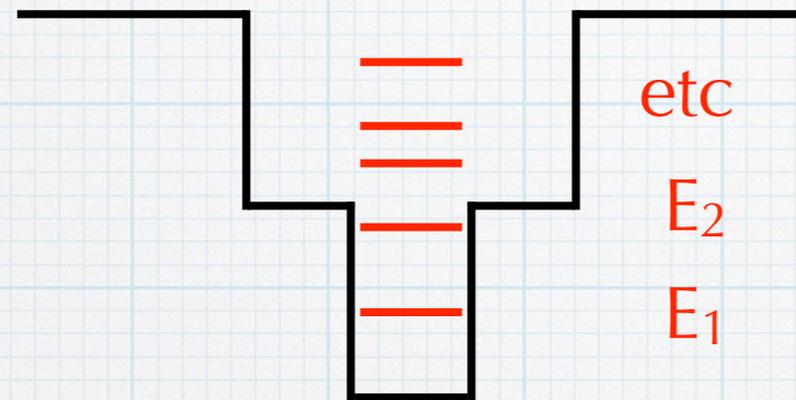
Finally, it can be used to obtain approximate solutions to complex potential profiles.
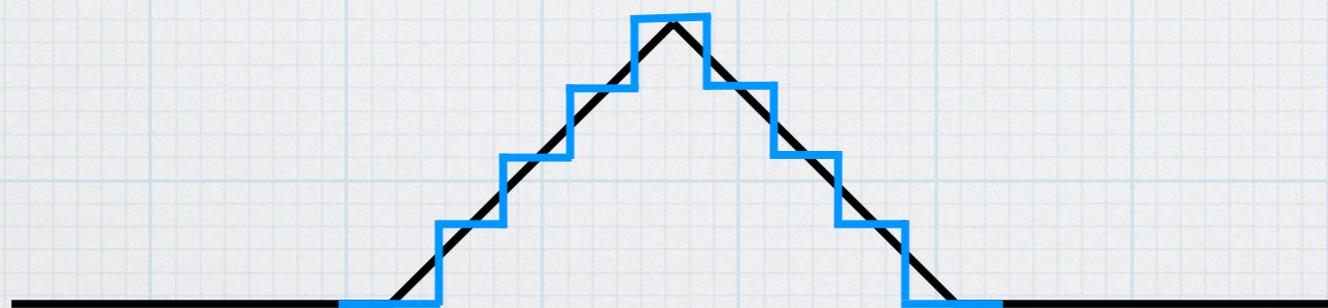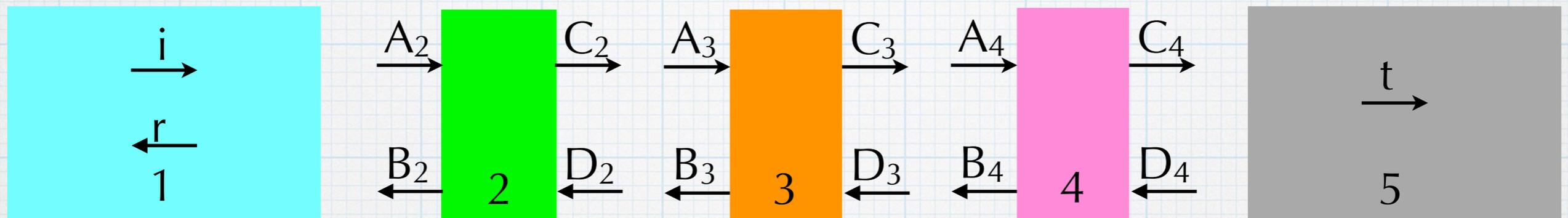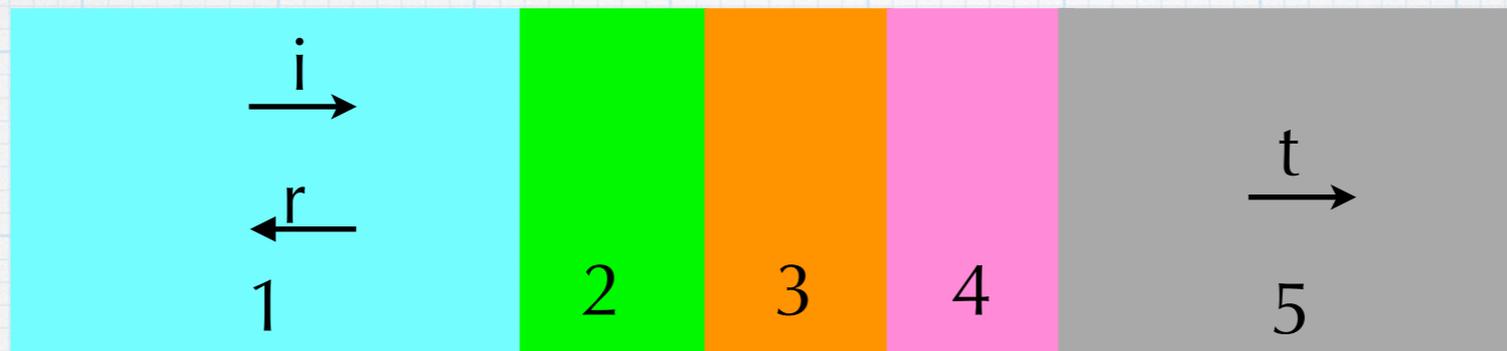
In a multi-layer problem, the difficulties come in handling the reflections at all of the interfaces. However, if we can determine how the plane waves relate from one side of a constant potential region to the other, including the effects of scattering at the interfaces, then we can relate the transmitted amplitude to the incident amplitude (or reflected to incident) of the overall system.

For each region, we'll try to write a matrix of the form:
$$\begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} = [M_n] \begin{bmatrix} C_n \\ D_n \end{bmatrix}$$

where $[M_n]$ is a 2x2 matrix describing the nth region.
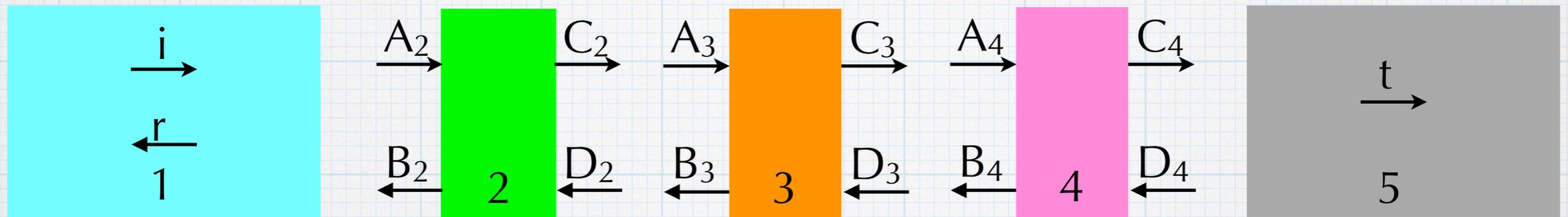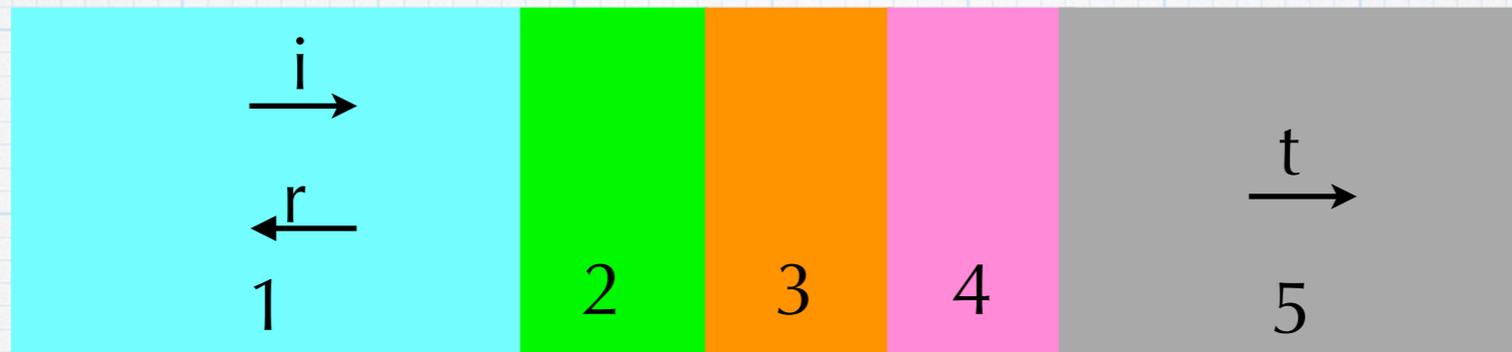
$$\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = [M_2] \begin{bmatrix} C_2 \\ D_2 \end{bmatrix} = [M_2] \begin{bmatrix} A_3 \\ B_3 \end{bmatrix}$$

$$= [M_2] [M_3] \begin{bmatrix} C_3 \\ D_3 \end{bmatrix} = [M_2] [M_3] \begin{bmatrix} A_4 \\ B_4 \end{bmatrix}$$

$$= [M_2] [M_3] [M_4] \begin{bmatrix} C_4 \\ D_4 \end{bmatrix}$$

$$\begin{bmatrix} i \\ r \end{bmatrix} = [I_L] [M_2] [M_3] [M_4] [I_R] \begin{bmatrix} t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i \\ r \end{bmatrix} = [I_L] [M_2] [M_3] [M_4] [I_R] \begin{bmatrix} t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i \\ r \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} t \\ 0 \end{bmatrix}$$

transmission: $\qquad \dfrac{t}{i} = \dfrac{1}{M_{11}}$

reflection: $\qquad \dfrac{r}{i} = \dfrac{M_{21}}{M_{11}}$

Finding reflection and transmission coefficients is a process of multiplying individual layer matrices to obtain an overall "system" matrix. The reflection and transmission coefficient ratios are found from the elements of the system matrix.

Unfortunately, our approach won't be quite as tidy as the previous slide would imply. Since the electron waves are scattered at <u>interfaces</u> between layers, we need a matrix to describe what happens at each interface. Of course, an interface is not a property of single layer, but depends on the layer properties on either side of the interface.

Also, the waves change phase as they propagate across regions where $E > U_n$, or they grow and decay exponentially across regions where $E < U_n$. We will need <u>propagation matrices</u> to describe these changes.

Intuitively then, each layer leads to two matrices to be included in the sequence, one propagation matrix and one interface matrix.

However, once we have the form of the propagation and interface matrices, we'll see that we can combine them in the right way to obtain a simple, one-matrix description of each layer.   However, getting to this point is not intuitive, so we'll take a round-about approach, but one that will hopefully give a clearer picture of what we're doing.

Interface matrices

Case 1 - E > U on both side of the interface.
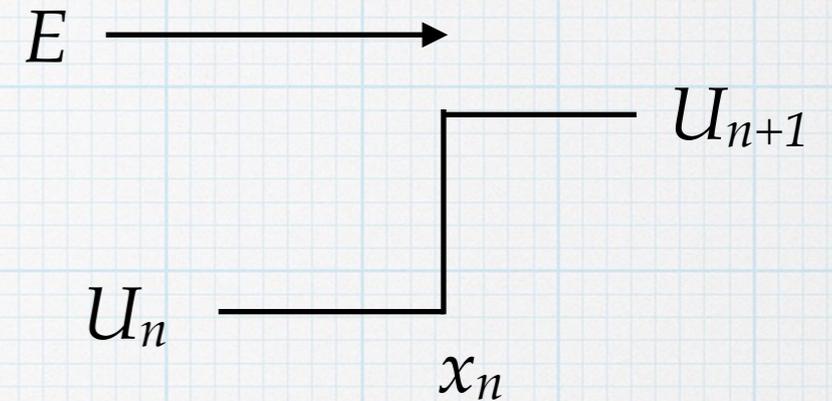
We assume the interface occurs at $x = x_n$. For $x < x_n$, the potential is U_n, and for $x > x_n$, it is U_{n+1}. Since $E > U_n$ and $E > U_{n+1}$, we use plane-wave solutions on both sides of the interface.

$$\psi_n(x) = A \exp[ik_n(x - x_n)] + B \exp[-ik_n(x - x_n)]$$

$$\psi_{n+1}(x) = C \exp[ik_{n+1}(x - x_n)] + D \exp[-ik_{n+1}(x - x_n)]$$

Apply the boundary conditions

$$A + B = C + D$$

$$ik_n A - ik_n B = ik_{n+1}C - ik_{n+1}D$$

Use the two equations to write A and B in terms of C and D.

$$A = \frac{1}{2}\left[1 + \frac{k_{n+1}}{k_n}\right] C + \frac{1}{2}\left[1 - \frac{k_{n+1}}{k_n}\right] D$$

$$B = \frac{1}{2}\left[1 - \frac{k_{n+1}}{k_n}\right] C + \frac{1}{2}\left[1 + \frac{k_{n+1}}{k_n}\right] D$$

In matrix form

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{k_{n+1}}{2k_n}\right) & \left(\frac{1}{2} - \frac{k_{n+1}}{2k_n}\right) \\ \left(\frac{1}{2} - \frac{k_{n+1}}{2k_n}\right) & \left(\frac{1}{2} + \frac{k_{n+1}}{2k_n}\right) \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$

Note that the matrix elements in this case are all real.

Case 2 - E > U on the left and E < U on the right.

As before, we assume the interface occurs at $x = x_n$. For $x < x_n$, the potential is U_n, and for $x > x_n$, it is U_{n+1}. For $x < x_n$, we need plane wave solutions and for $x > x_n$, we use growing and decaying exponentials.

U_{n+1}

$E \longrightarrow$

U_n

x_n
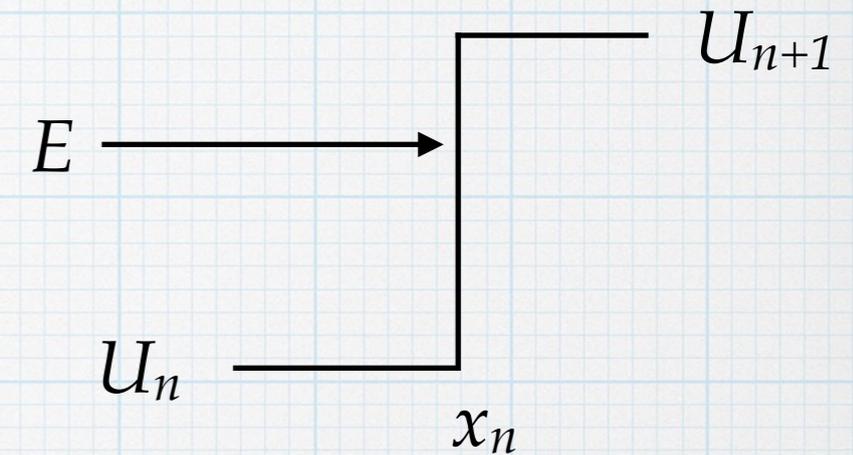
$$\psi_n(x) = A \exp\left[ik_n(x - x_n)\right] + B \exp\left[-ik_n(x - x_n)\right]$$

$$\psi_{n+1}(x) = C \exp\left[\alpha_{n+1}(x - x_n)\right] + D \exp\left[-\alpha_{n+1}(x - x_n)\right]$$

Using the connection rules at the interface:

$$A + B = C + D$$

$$ik_n A - ik_n B = \alpha_{n+1}C - \alpha_{n+1}D$$

Solving for A_n and B_n in terms of C_{n+1} and D_{n+1}:

$$A_n = \frac{1}{2}\left[1 - i\frac{\alpha_{n+1}}{k_n}\right]C_{n+1} + \frac{1}{2}\left[1 + i\frac{\alpha_{n+1}}{k_n}\right]D_{n+1}$$

$$B_n = \frac{1}{2}\left[1 + i\frac{\alpha_{n+1}}{k_n}\right]C_{n+1} + \frac{1}{2}\left[1 - i\frac{\alpha_{n+1}}{k_n}\right]D_{n+1}$$

In matrix form

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - i\frac{\alpha_{n+1}}{2k_n}\right) & \left(\frac{1}{2} + i\frac{\alpha_{n+1}}{2k_n}\right) \\ \left(\frac{1}{2} + i\frac{\alpha_{n+1}}{2k_n}\right) & \left(\frac{1}{2} - i\frac{\alpha_{n+1}}{2k_n}\right) \end{bmatrix} \begin{bmatrix} C_{n+1} \\ D_{n+1} \end{bmatrix}$$

Note that the matrix elements in this case are all complex.

Case 3 - E < U on the left and E > U on the right.

Same song, third verse.
For $x < x_n$, the potential is U_n, and for $x > x_n$, it is U_{n+1}. For $x < x_n$, we need growing and decaying exponentials and for $x > x_n$, we use plane wave solutions.
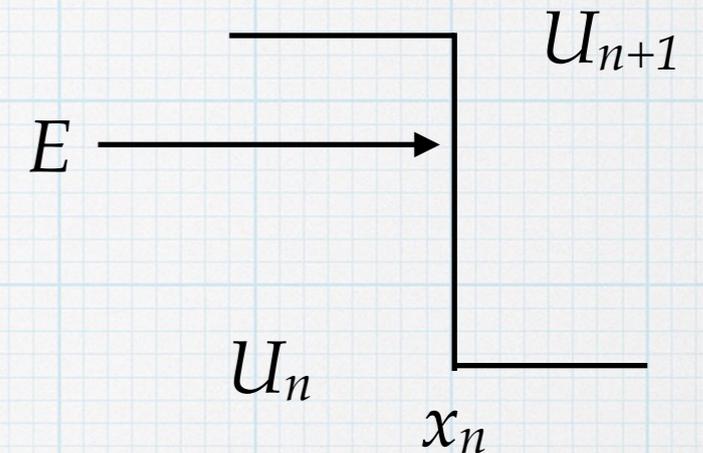
$$\psi_n(x) = A_n \exp\left[\alpha(x - x_n)\right] + B_n \exp\left[-\alpha(x - x_n)\right]$$

$$\psi_{n+1}(x) = C_{n+1} \exp\left[ik_{n+1}(x - x_n)\right] + D_{n+1} \exp\left[-k_{n+1}(x - x_n)\right]$$

Using the connection rules at the interface:

$$A_n + B_n = C_{n+1} + D_{n+1}$$

$$\alpha_n A_n - \alpha_n B_n = ik_{n+1}C_{n+1} - ik_{n+1}D_{n+1}$$

Solving for A_n and B_n in terms of C_{n+1} and D_{n+1}:

$$A_n = \frac{1}{2}\left[1 + i\frac{k_{n+1}}{\alpha_n}\right]C_{n+1} + \frac{1}{2}\left[1 - i\frac{k_{n+1}}{\alpha_n}\right]D_{n+1}$$

$$B_n = \frac{1}{2}\left[1 - i\frac{k_{n+1}}{\alpha_n}\right]C_{n+1} + \frac{1}{2}\left[1 + i\frac{k_{n+1}}{\alpha_n}\right]D_{n+1}$$

In matrix form

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + i\frac{k_{n+1}}{2\alpha_n}\right) & \left(\frac{1}{2} - i\frac{k_{n+1}}{2\alpha_n}\right) \\ \left(\frac{1}{2} - i\frac{k_{n+1}}{2\alpha_n}\right) & \left(\frac{1}{2} + i\frac{k_{n+1}}{2\alpha_n}\right) \end{bmatrix} \begin{bmatrix} C_{n+1} \\ D_{n+1} \end{bmatrix}$$

The matrix elements are again all complex.

Case 4 - E < U on the left and E < U on the right.

U_n ⎯⎯⎯ U_{n+1}

Final stanza. Everything is growing and decaying exponentials.

E ⟶

x_n
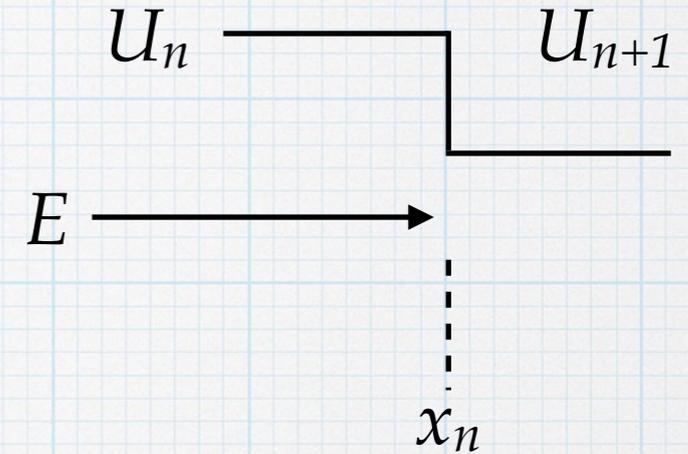
$$\psi_n(x) = A_n \exp\left[\alpha_n(x - x_n)\right] + B_n \exp\left[-\alpha_n(x - x_n)\right]$$

$$\psi_{n+1}(x) = C_{n+1} \exp\left[\alpha_{n+1}(x - x_n)\right] + D_{n+1} \exp\left[-\alpha_{n+1}(x - x_n)\right]$$

Applying the boundary conditions one last time:

$$A_n + B_n = C_{n+1} + D_{n+1}$$

$$\alpha_n A_n - \alpha_n B_n = \alpha_{n+1} C_{n+1} - \alpha_{n+1} D_{n+1}$$

Solving for A_n and B_n in terms of C_{n+1} and D_{n+1}:

$$A_n = \frac{1}{2}\left[1 + \frac{\alpha_{n+1}}{\alpha_n}\right]C_{n+1} + \frac{1}{2}\left[1 - \frac{\alpha_{n+1}}{\alpha_n}\right]D_{n+1}$$

$$B_n = \frac{1}{2}\left[1 - \frac{\alpha_{n+1}}{\alpha_n}\right]C_{n+1} + \frac{1}{2}\left[1 + \frac{\alpha_{n+1}}{\alpha_n}\right]D_{n+1}$$

In matrix form

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{\alpha_{n+1}}{2\alpha_n}\right) & \left(\frac{1}{2} - \frac{\alpha_{n+1}}{2\alpha_n}\right) \\ \left(\frac{1}{2} - \frac{\alpha_{n+1}}{2\alpha_n}\right) & \left(\frac{1}{2} + \frac{\alpha_{n+1}}{2\alpha_n}\right) \end{bmatrix} \begin{bmatrix} C_{n+1} \\ D_{n+1} \end{bmatrix}$$

The matrix elements are again all real.

Summary:

$$\begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}$$

$E > U_n$
$E > U_{n+1}$

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{k_{n+1}}{2k_n} \right) & \left(\frac{1}{2} - \frac{k_{n+1}}{2k_n} \right) \\ \left(\frac{1}{2} - \frac{k_{n+1}}{2k_n} \right) & \left(\frac{1}{2} + \frac{k_{n+1}}{2k_n} \right) \end{bmatrix}$$

$E > U_n$
$E < U_{n+1}$

$$\begin{bmatrix} \left(\frac{1}{2} - i\frac{\alpha_{n+1}}{2k_n} \right) & \left(\frac{1}{2} + i\frac{\alpha_{n+1}}{2k_n} \right) \\ \left(\frac{1}{2} + i\frac{\alpha_{n+1}}{2k_n} \right) & \left(\frac{1}{2} - i\frac{\alpha_{n+1}}{2k_n} \right) \end{bmatrix}$$

$E < U_n$
$E > U_{n+1}$

$$\begin{bmatrix} \left(\frac{1}{2} + i\frac{k_{n+1}}{2\alpha_n} \right) & \left(\frac{1}{2} - i\frac{k_{n+1}}{2\alpha_n} \right) \\ \left(\frac{1}{2} - i\frac{k_{n+1}}{2\alpha_n} \right) & \left(\frac{1}{2} + i\frac{k_{n+1}}{2\alpha_n} \right) \end{bmatrix}$$

$E < U_n$
$E < U_{n+1}$

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{\alpha_{n+1}}{2\alpha_n} \right) & \left(\frac{1}{2} - \frac{\alpha_{n+1}}{2\alpha_n} \right) \\ \left(\frac{1}{2} - \frac{\alpha_{n+1}}{2\alpha_n} \right) & \left(\frac{1}{2} + \frac{\alpha_{n+1}}{2\alpha_n} \right) \end{bmatrix}$$

It's pretty easy to see that k should be replaced with $-i\alpha$ in regions where $E < U$.

Propagation matrices

As a traveling wave crosses a region where $E > U_n$, it changes phase.

For a wave traveling in the $+x$ direction, we can write by inspection:

$$C = A \exp(ik_nL_n)$$

For a wave traveling the $-x$ direction

$$D = B \exp(-ik_nL_n)$$

Expressed in matrix form:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \exp(-ik_nL_n) & 0 \\ 0 & \exp(ik_nL_n) \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$
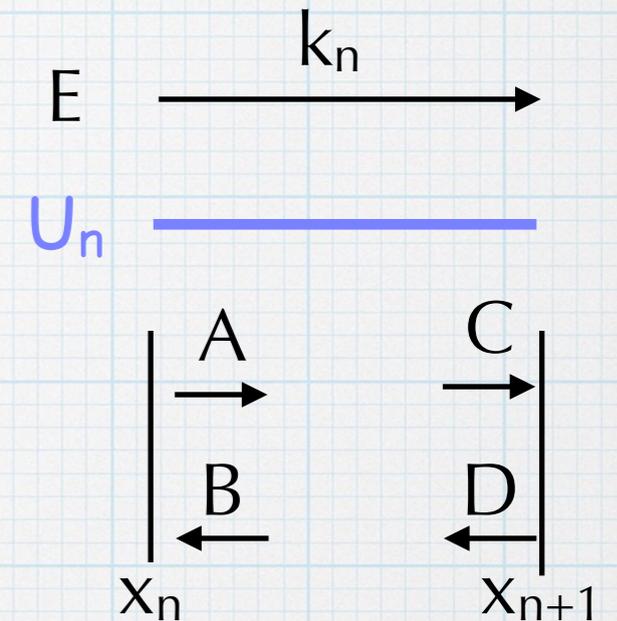
For an evanescent wave in a region where $E < U_n$, there is no phase change, but the amplitude must change exponentially.

$$C = A \exp(\alpha_n L_n)$$

$$D = B \exp(-\alpha_n L_n)$$
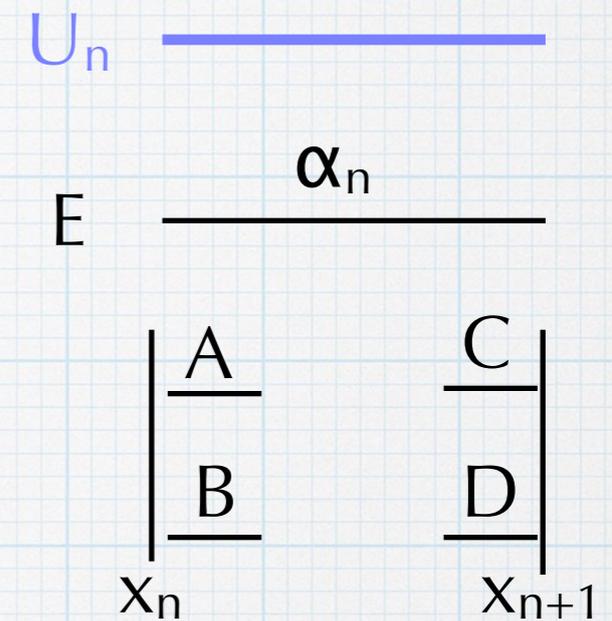
In this case, the "propagation" matrix is

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \exp(-\alpha_n L_n) & 0 \\ 0 & \exp(\alpha_n L_n) \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$
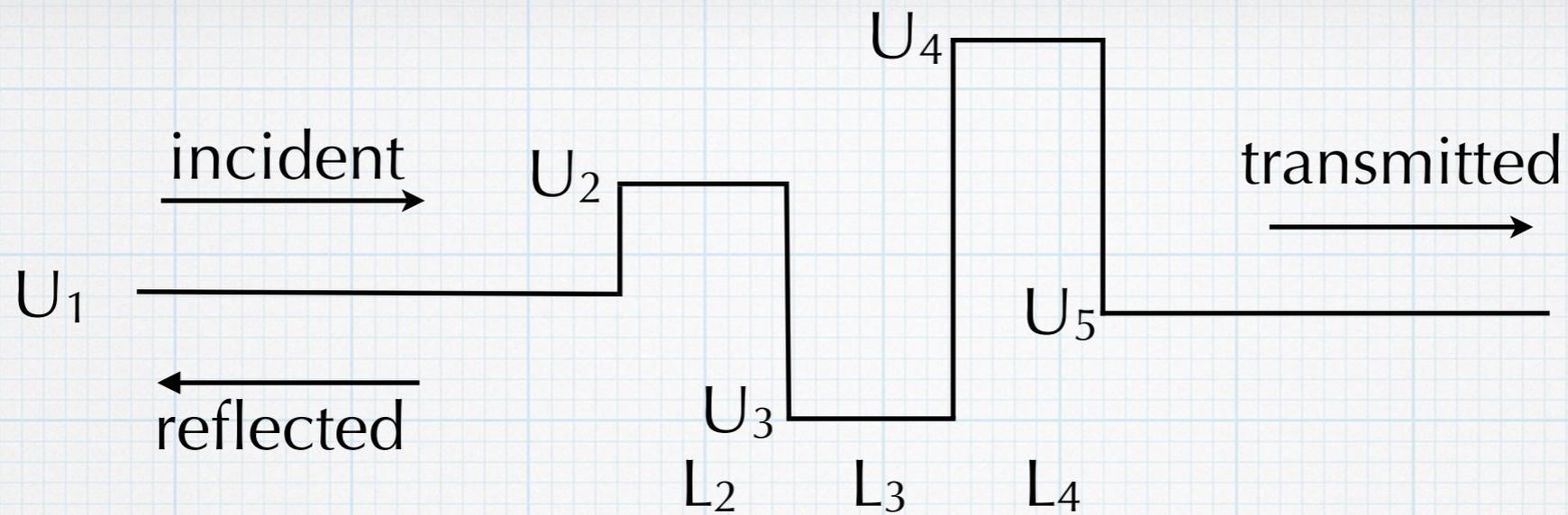
Looking again at the potential posed on the first slide, we see that we need a whole string of interface and propagation matrices.

$$\begin{bmatrix} i \\ r \end{bmatrix} = [I_{12}] \, [P_2] \, [I_{23}] \, [P_3] \, [I_{34}] \, [P_4] \, [I_{45}] \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} t \\ 0 \end{bmatrix}$$

$$\frac{t}{i} = \frac{1}{M_{11}}$$

$$T = \frac{k_5}{k_1} \frac{|t|^2}{|i|^2} = \frac{k_5}{k_1} \frac{1}{|M_{11}|^2}$$

$$\frac{r}{i} = \frac{M_{21}}{M_{11}}$$

$$R = \frac{k_1}{k_1} \frac{|r|^2}{|i|^2} = \frac{|M_{21}|^2}{|M_{11}|^2}$$

The matrix technique lends itself well to programming in Matlab or some other language. However, handling the interfaces is a bit unwieldy since the interface matrix involve properties of two layers. It would be nice if everything about a given layer could be included in one matrix. Can this be done? Look at the form of an interface matrix.

$$I_{n,n+1} = \frac{1}{2} \begin{bmatrix} \left(1 + \frac{k_{n+1}}{k_n}\right) & \left(1 - \frac{k_{n+1}}{k_n}\right) \\ \left(1 - \frac{k_{n+1}}{k_n}\right) & \left(1 + \frac{k_{n+1}}{k_n}\right) \end{bmatrix}$$

Mathematically, it can be split in two:

$$I_{n,n+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \left(\frac{1}{k_n}\right) \\ 1 & \left(-\frac{1}{k_n}\right) \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ (k_{n+1}) & (-k_{n+1}) \end{bmatrix}$$

The matrix with k_n represents the left side of the interface. The matrix with k_{n+1} represents the right side of the interface.

$$[I_{n,n+1}] = [L_n] [R_{n+1}]$$

Alternatively, the matrix with k_n represents the right <u>end</u> of the nth layer. The matrix with k_{n+1} represents the left <u>end</u> of the (n+1)st layer.

Look a section of the sequence of matrices from our original problem.

$$[M] = [I_{12}] [P_2] [I_{23}] [P_3] [I_{34}] [P_4] [I_{45}]$$

Split the interface matrices

$$[M] = [L_1] [R_2] [P_2] [L_2] [R_3] [P_3] [L_3] [R_4] [P_4] [L_4] [R_5]$$

$$= [L_1] [R_2] [P_2] [L_2] [R_3] [P_3] [L_3] [R_4] [P_4] [L_4] [R_5]$$

$$= [L_1] [M_2] [M_3] [M_4] [R_5]$$

where

$$[M_n] = [R_n] [P_n] [L_n]$$

The layer matrix $[M_n]$ contains all of the information about a particular layer. The parameters for layer n show up only in that particular matrix. This makes it easier to specify and compute the matrices in a program.

$$[M_n] = [R_n] [P_n] [L_n]$$

In the layer, if $E > U_n$ (propagating wave)

$$[M_n] = \begin{bmatrix} \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right) \\ \left(\frac{k_n}{\sqrt{2}}\right) & \left(-\frac{k_n}{\sqrt{2}}\right) \end{bmatrix} \cdot \begin{bmatrix} \exp\left(-ik_nL_n\right) & 0 \\ 0 & \exp\left(ik_nL_n\right) \end{bmatrix} \cdot \begin{bmatrix} \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}k_n}\right) \\ \left(\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}k_n}\right) \end{bmatrix}$$

$$[M_n] = \begin{bmatrix} \cos\left(k_nL_n\right) & -\frac{i}{k_n}\sin\left(k_nL_n\right) \\ -ik_n\sin\left(k_nL_n\right) & \cos\left(k_nL_n\right) \end{bmatrix}$$

If $E < U_n$ (evanescent wave)

$$[M_n] = \begin{bmatrix} \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right) \\ \left(-\frac{i\alpha_n}{\sqrt{2}}\right) & \left(\frac{i\alpha_n}{\sqrt{2}}\right) \end{bmatrix} \cdot \begin{bmatrix} \exp\left(-i\alpha_nL_n\right) & 0 \\ 0 & \exp\left(\alpha_nL_n\right) \end{bmatrix} \cdot \begin{bmatrix} \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{i}{\sqrt{2}\alpha_n}\right) \\ \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{-i}{\sqrt{2}\alpha_n}\right) \end{bmatrix}$$

$$[M_n] = \begin{bmatrix} \cosh\left(\alpha_nL_n\right) & \frac{i}{\alpha_n}\sinh\left(\alpha_nL_n\right) \\ -i\alpha_n\sinh\left(\alpha_nL_n\right) & \cosh\left(\alpha_nL_n\right) \end{bmatrix}$$

Example - tunneling through a square barrier (redux)

We've done this before and know the result. This may a good test for our matrix approach.

incident $\longrightarrow$

$U = U_o$

$\longrightarrow$ transmitted

reflected $\longleftarrow$

$U = 0$

$x = 0 \quad x = L$

① ② ③
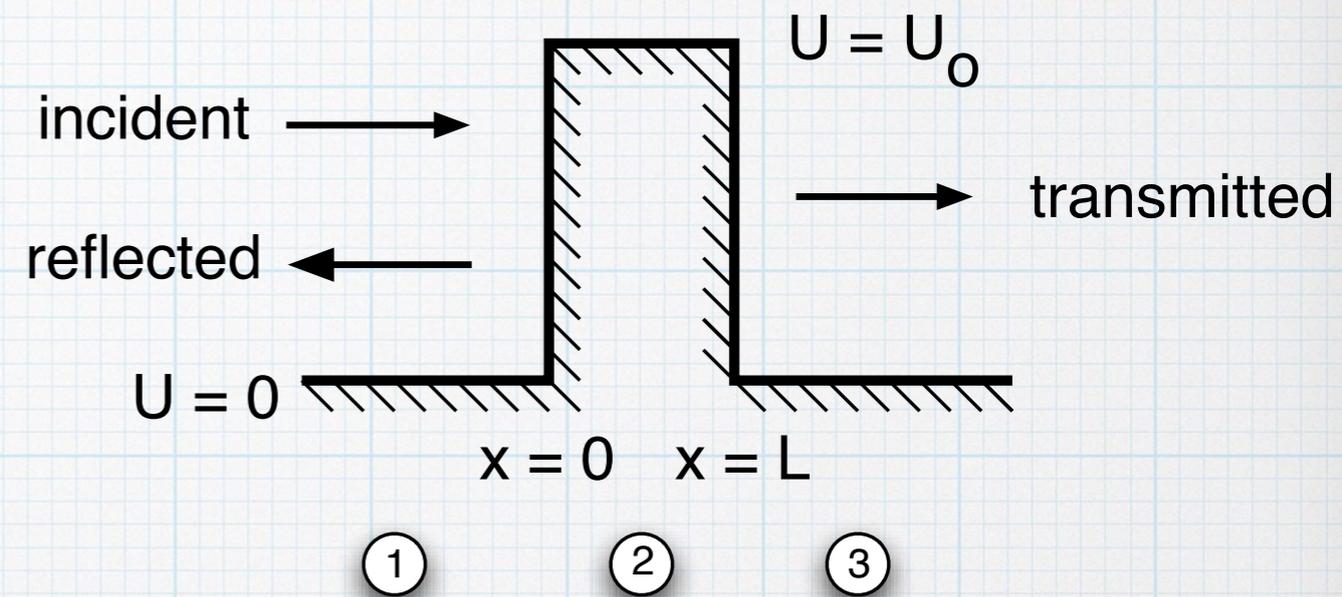
There is an electron incident from the left in region 1 (where $U = 0$), so we need a left half matrix for region 1 at $x = 0$. We need a layer matrix (of the $E < U$ variety) for the barrier. Finally, we must have a right-half matrix for region 3 at $x = L$. Since $k_1 = k_3 = k$ and region 2 is characterized by α, we can dispense with the subscripts.

$$[M] = \begin{bmatrix} \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2k}}\right) \\ \left(\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2k}}\right) \end{bmatrix} \cdot \begin{bmatrix} \cosh(\alpha L) & \frac{i}{\alpha}\sinh(\alpha L) \\ -i\alpha\sinh(\alpha L) & \cosh(\alpha L) \end{bmatrix} \cdot \begin{bmatrix} \left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right) \\ \left(-\frac{k}{\sqrt{2}}\right) & \left(-\frac{k}{\sqrt{2}}\right) \end{bmatrix}$$

Now comes tedious algebra to get to the answer. Note that to the find the transmission probability, we only need M_{11}.

$$M_{11} = \cosh\left(\alpha L\right) + \frac{i}{2}\left(\frac{k}{\alpha} - \frac{\alpha}{k}\right)\sinh\left(\alpha L\right)$$

$$|M_{11}|^2 = \cosh^2\left(\alpha L\right) + \frac{1}{4}\left(\frac{k}{\alpha} - \frac{\alpha}{k}\right)^2\sinh^2\left(\alpha L\right)$$

$$= 1 + \sinh^2\left(\alpha L\right) + \frac{1}{4}\left[\left(\frac{k}{\alpha}\right)^2 - 2 + \left(\frac{\alpha}{k}\right)^2\right]\sinh^2\left(\alpha L\right)$$

$$= 1 + \frac{1}{4}\left[\left(\frac{k}{\alpha}\right)^2 + 2 + \left(\frac{\alpha}{k}\right)^2\right]\sinh^2\left(\alpha L\right)$$

$$= 1 + \frac{1}{4}\left(\frac{k}{\alpha} + \frac{\alpha}{k}\right)^2\sinh^2\left(\alpha L\right)$$

$$= 1 + \left[\frac{U_o^2}{4E\left(U_o - E\right)}\right]^2\sinh^2\left(\alpha L\right)$$

$$T = \frac{k_3}{k_1}\frac{1}{|M_{11}|^2} = \frac{1}{|M_{11}|^2}$$

As we saw earlier when we first looked at tunneling.

$$= \left\{1 + \left[\frac{U_o^2}{4E\left(U_o - E\right)}\right]^2\sinh^2\left(\alpha L\right)\right\}^{-1} \approx \left[\frac{16E\left(U_o - E\right)}{U_o^2}\right]\exp\left(-2\alpha L\right)$$

Bound states

Can the matrix method be used to learn something about bound states?

It requires a slightly different approach, since a bound state does not "propagate" and so we will not calculate transmission or reflection probabilities.

A bound state is characterized by the requirement that
$\psi(\, x \to \pm\infty\,) \to 0$.

This requirement means that, in the "input" and "output" regions, the wave function must be in the form of a decaying exponential.

$$\begin{bmatrix} 0 \\ B \end{bmatrix} = [M] \begin{bmatrix} C \\ 0 \end{bmatrix}$$

$$M_{11} = 0$$

So the matrix procedure would be to find the total matrix description for the problem, and then finds the roots of the M_{11} matrix element.

Example - finite height square well (redux)

$$U = U_o$$

$$U = U_o$$

②   ①   ③

$$-L/2$$ $$+L/2$$

$$U = 0$$

$$x$$

$$0$$
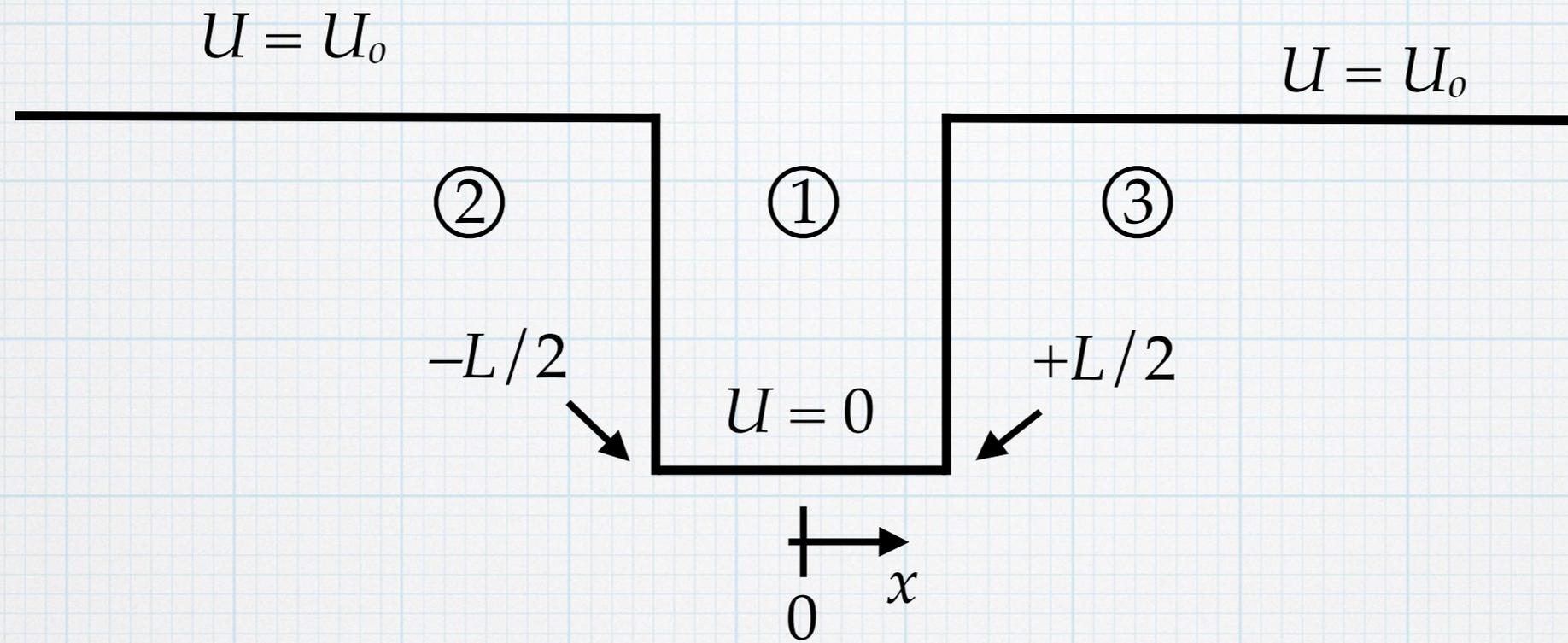
To give a quantitative comparison, use $U_o = 1$ eV and $L = 2$ nm.

Using the even / odd approach with solving the transcendental characteristic equation repeatedly give four solutions:

$$w_1 \approx 1.31 \rightarrow E_1 = 0.066 \text{ eV} \qquad w_2 \approx 2.608 \rightarrow E_2 = 0.260 \text{ eV}$$

$$w_3 \approx 3.86 \rightarrow E_3 = 0.569 \text{ eV} \qquad w_4 \approx 4.96 \rightarrow E_3 = 0.939 \text{ eV}$$

$U = U_o$

$U = U_o$

② ① ③

$-L/2$ $+L/2$

$U = 0$

0 x
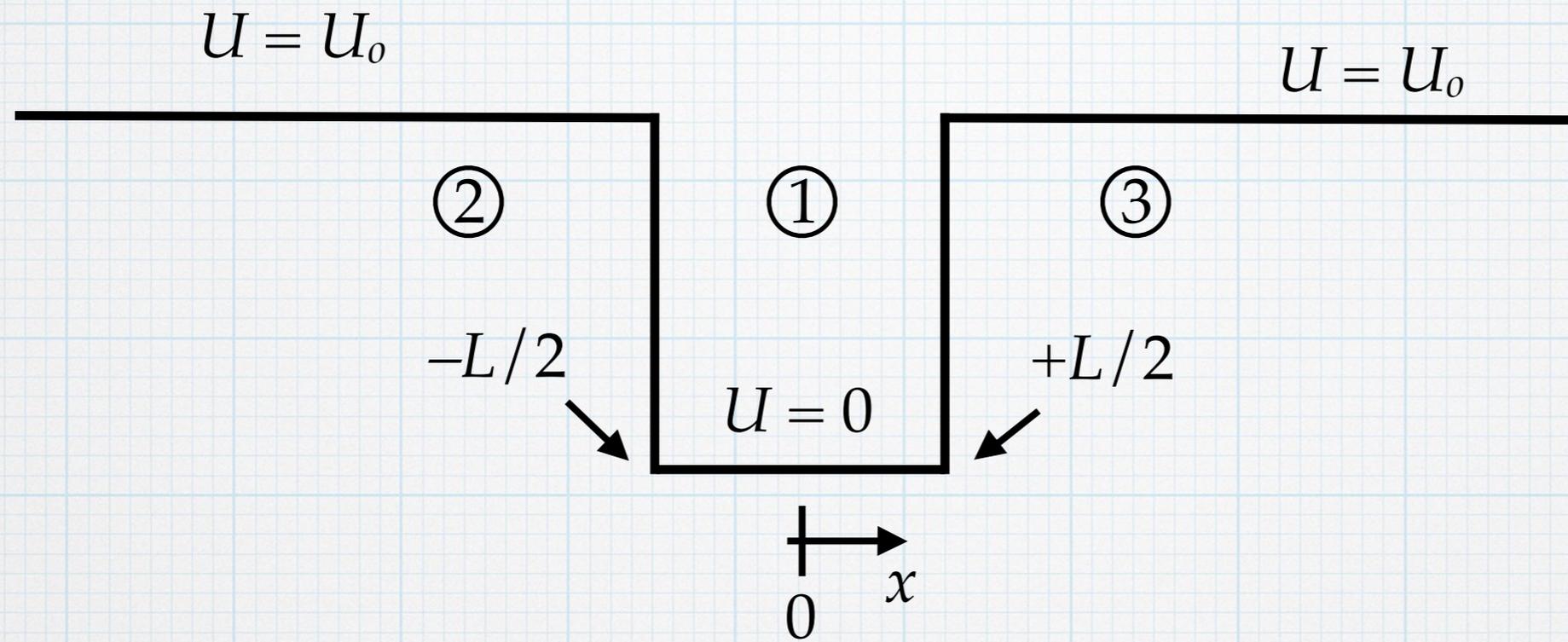
$$[M_T] = [L_2] [M_1] [R_3]$$

$$= \begin{bmatrix} \left(\dfrac{1}{\sqrt{2}}\right) & \left(\dfrac{i}{\sqrt{2}\alpha}\right) \\ \left(\dfrac{1}{\sqrt{2}}\right) & \left(-\dfrac{i}{\sqrt{2}\alpha}\right) \end{bmatrix} \cdot \begin{bmatrix} \cos(kL) & \dfrac{-i}{k}\sin(kL) \\ -ik\sin(kL) & \cos(kL) \end{bmatrix} \cdot \begin{bmatrix} \left(\dfrac{1}{\sqrt{2}}\right) & \left(\dfrac{1}{\sqrt{2}}\right) \\ \left(\dfrac{-i\alpha}{\sqrt{2}}\right) & \left(\dfrac{i\alpha}{\sqrt{2}}\right) \end{bmatrix}$$

$$M_{11} = \cos(kL) + \frac{1}{2}\left[\frac{\alpha}{k} - \frac{k}{\alpha}\right]\sin(kL)$$

To find the bound states, set $M_{11} = 0$ and find the roots.

$$\cos(kL) + \frac{1}{2}\left[\frac{\alpha}{k} - \frac{k}{\alpha}\right]\sin(kL) = 0$$

Of course, k and α depend on energy, so we will be finding particular energies for which the above equation goes to 0. An easy way to see what is going on is to make a plot.
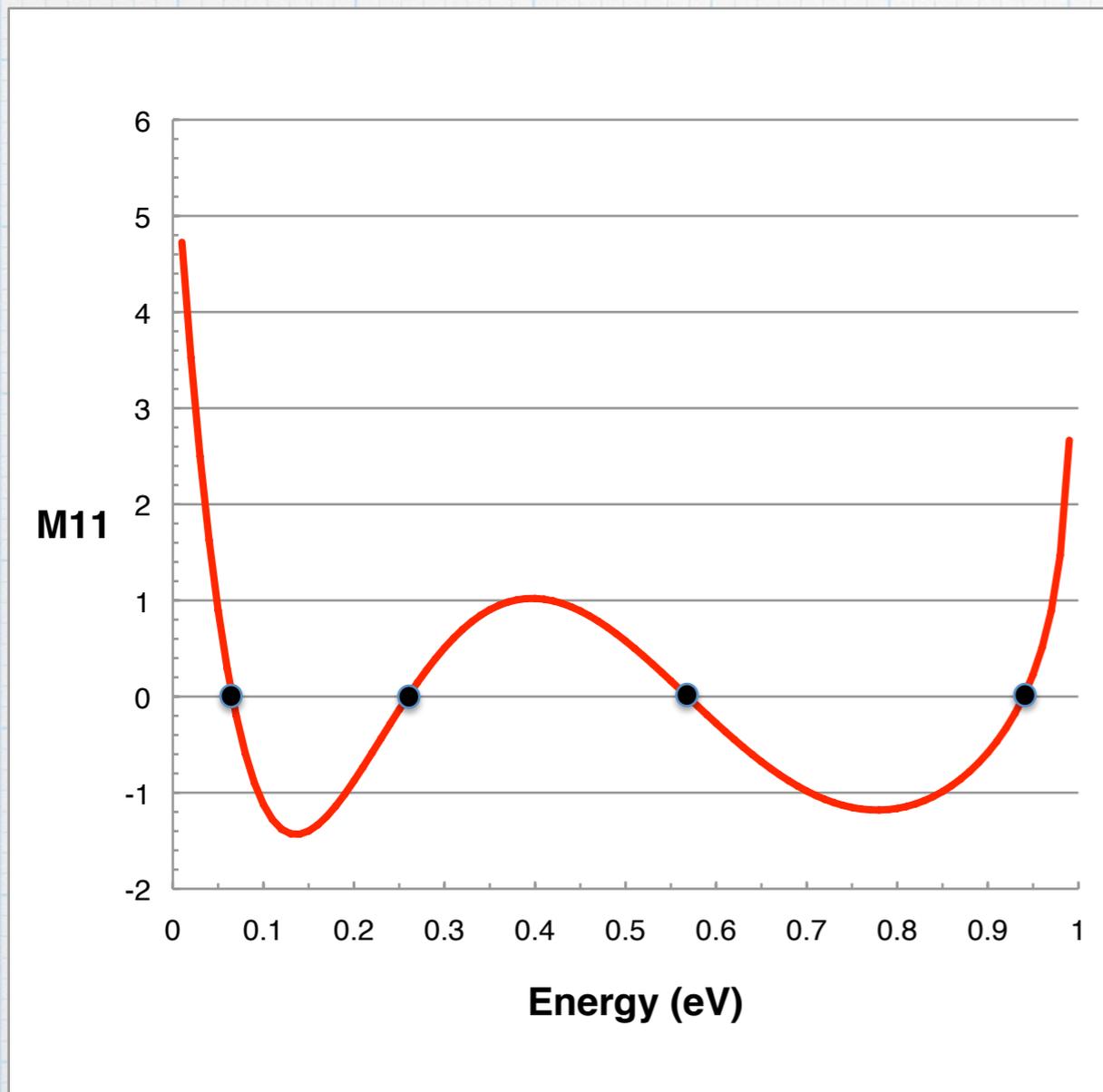
Just from the plot, we see that the approximate energies are:

$E_1 \approx 0.06$ eV

$E_2 \approx 0.25$ eV

$E_3 \approx 0.56$ eV

$E_4 \approx 0.94$ eV

With just a bit of effort, the numbers can be made more precise.