Fourier series

A Fourier series can be used to express any periodic function in terms of a series of cosines and sines.

A periodic function, defined by a period T,

$$v(t + T) = v(t)$$

Familiar periodic functions: square, triangle, sawtooth, and sinusoids (of course). A DC voltage or current also fits the definition of a periodic function.

According to Fourier, a periodic function $v(t)$ can be written as:

$$v(t) = a_o + a_1 \cos \omega_o t + a_2 \cos 2\omega_o t + a_3 \cos 3\omega_o t + \ldots$$

$$+ b_1 \sin \omega_o t + b_2 \sin 2\omega_o t + b_3 \sin 3\omega_o t + \ldots$$

$$= a_o + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t) \right]$$

$$\omega_o = 2\pi f_o = \frac{2\pi}{T}$$

$$v(t) = a_o + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t) \right]$$

$$\omega_o = 2\pi f_o = \frac{2\pi}{T}$$

The fundamental frequency f_o (or ωo_0), is defined by the period of the original function. The $n = 1$ terms are the fundamental terms. The $n = 2$, 3, 4, and higher terms are the *harmonics* of the fundamental.

Expressing a particular periodic function in terms of a Fourier series is "simply" a matter of picking the appropriate coefficients a_n and b_n to use for each term.

Mathematical legalese:

1. $f(t)$ is single-valued everywhere. (i.e. It is a "good" function.)

2. The integral $\int_{t_o}^{t_o+T} |f(t)|\,dt$ exists for any t_o. (i.e. It is not infinite.)

3. $f(t)$ has a finite number of discontinuities within any one period.

4. $f(t)$ has a finite number of maxima and minima in any one period.

Finding a_n and b_n

Determining the coefficients is straight-forward, once we recall some special properties of sinusoids. Below are a few preliminary insights that will be helpful in the calculations that follow. In the following relations, *m* and *n* are integers. (You should confirm these results.)

$$\int_0^T \sin\left(n\frac{2\pi}{T}\right) dt = 0 \qquad \int_0^T \cos\left(n\frac{2\pi}{T}\right) dt = 0$$

$$\int_0^T \sin\left(m\omega_o t\right)\cos\left(n\omega_o t\right) dt = 0$$

$$\int_0^T \sin\left(m\omega_o t\right)\sin\left(n\omega_o t\right) dt = 0 \qquad m \neq n$$

$$\int_0^T \cos\left(m\omega_o t\right)\cos\left(n\omega_o t\right) dt = 0 \qquad m \neq n$$

$$\int_0^T \sin^2\left(n\omega_o t\right) dt = \frac{T}{2} \qquad \int_0^T \cos^2\left(n\omega_o t\right) dt = \frac{T}{2}$$

$$v(t) = a_o + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t) \right]$$

Start by taking the integral over one period for both sides of the equation:

$$\int_0^T v(t)\,dt = \int_0^T a_o dt + \int_0^T \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t) \right] dt$$

$$\int_0^T v(t)\,dt = \int_0^T a_o dt + \sum_{n=1}^{\infty} \left[a_n \int_0^T \cos(n\omega_o t)\,dt + b_n \int_0^T \sin(n\omega_o t)\,dt \right]$$

The integral over every term in the summation will be zero, leaving only the one term involving ao.

$$\int_0^T v(t)\,dt = \int_o^T a_o dt = a_o T$$

$$a_o = \frac{1}{T} \int_0^T v(t)\,dt$$

We see that a_o is simply the average, or DC, component of $v(t)$.

$$v(t) = a_o + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t) \right]$$

Finding a_n and b_n coefficients involves similar mathematical shenanigans. To find the a_n coefficients, multiply both sides of the Fourier expression by $\cos(m\omega_0 t)$, where m is some specific integer and then integrate over one period.

$$\int_0^T v(t) \cos(m\omega_o t)\, dt = \int_0^T a_o \cos(m\omega_o t)\, dt$$

$$+ \int_0^T \left[\sum_{n=1}^{\infty} a_n \cos(n\omega_o t) \cos(m\omega_o t) \right] dt$$

$$+ \int_0^T \left[\sum_{n=1}^{\infty} b_n \sin(n\omega_o t) \cos(m\omega_o t) \right] dt$$

$$\int_0^T v(t)\cos(m\omega_o t)\,dt = a_o\int_0^T \cos(m\omega_o t)\,dt$$

$$+\sum_{n=1}^{\infty} a_n\left[\int_0^T \cos(n\omega_o t)\cos(m\omega_o t)\,dt\right]$$

$$+\sum_{n=1}^{\infty} b_n\left[\int_0^T \sin(n\omega_o t)\cos(m\omega_o t)\,dt\right]$$

We see that the first term is zero. As we examine the two summations we note that every single term in the second one will be zero and in the first one all terms will be zero, *except* for the single where $m = n$. So the equation with two infinitely long summations boils down to

$$\int_0^T v(t)\cos(n\omega_o t)\,dt = a_n\int_0^T \cos^2(n\omega_o t)\,dt = a_n\frac{T}{2}$$

$$a_n = \frac{2}{T}\int_0^T v(t)\cos(n\omega_o t)\,dt$$

To find the coefficients for the sine terms, we use the same trick, except that we multiply by $\sin(m\omega_0 t)$ before integrating.

$$\int_0^T v(t)\sin\left(m\omega_o t\right)dt = \int_0^T a_o \sin\left(m\omega_o t\right)dt$$

$$+\int_0^T \left[\sum_{n=1}^{\infty} a_n \cos\left(n\omega_o t\right)\sin\left(m\omega_o t\right)\right]dt$$

$$+\int_0^T \left[\sum_{n=1}^{\infty} b_n \sin\left(n\omega_o t\right)\sin\left(m\omega_o t\right)\right]dt$$

$$= a_o \int_0^T \sin\left(m\omega_o t\right)dt$$

$$+\sum_{n=1}^{\infty} a_n \left[\int_0^T \cos\left(n\omega_o t\right)\sin\left(m\omega_o t\right)dt\right]$$

$$+\sum_{n=1}^{\infty} b_n \left[\int_0^T \sin\left(n\omega_o t\right)\sin\left(m\omega_o t\right)dt\right]$$

Once again, every single term integrates to zero, except for the the only that has two sines with $m = n$.

$$\int_0^T v(t)\sin\left(n\omega_o t\right) dt = b_n \int_0^T \sin^2\left(n\omega_o t\right) dt = b_n \frac{T}{2}$$

$$b_n = \frac{2}{T}\int_0^T v(t)\sin\left(n\omega_o t\right) dt$$

So, every periodic function can be represented by its Fourier series

$$v(t) = a_o + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\omega_o t\right) + b_n \sin\left(n\omega_o t\right)\right]$$

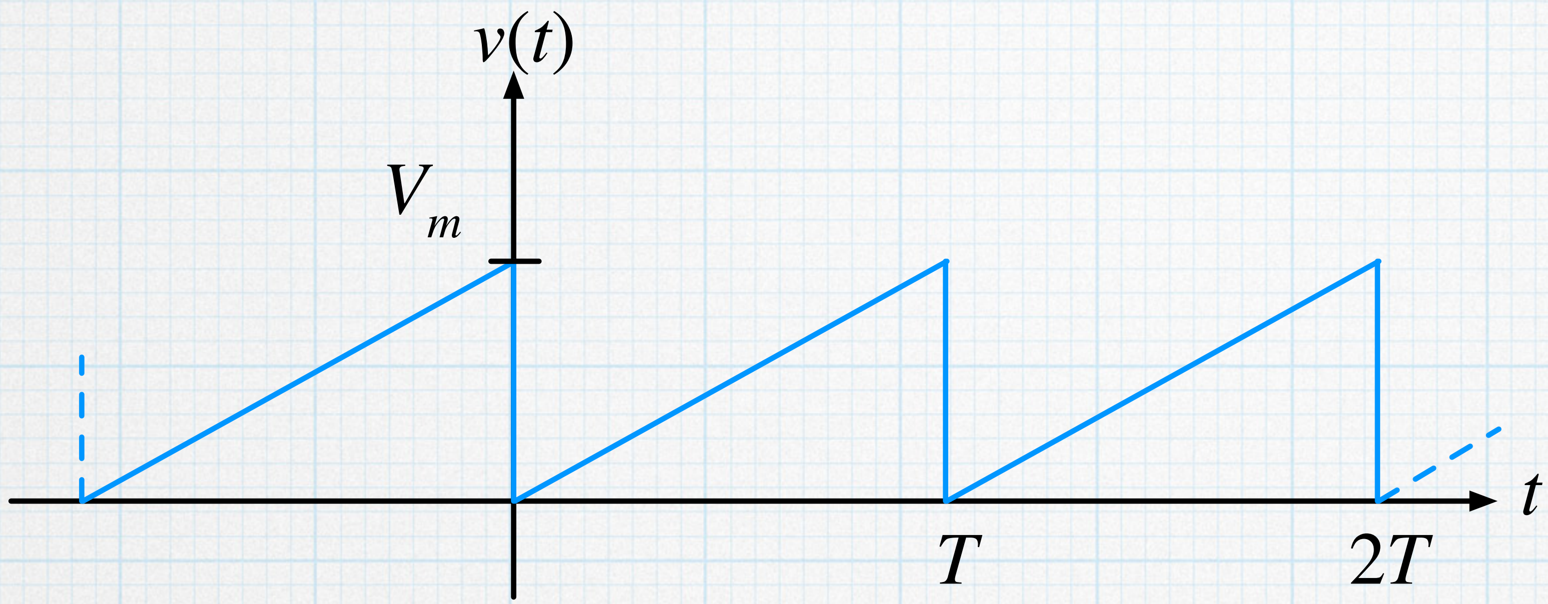
$$a_n = \frac{2}{T}\int_0^T v(t)\cos\left(n\omega_o t\right) dt \qquad b_n = \frac{2}{T}\int_0^T v(t)\sin\left(n\omega_o t\right) dt$$

$$a_o = \frac{1}{T}\int_0^T v(t)\, dt$$

(Note that a_o is just a special case of the a_n calculation.)

Everything is just sines and cosines! What a concept.

Example - sawtooth (ramp)



$$v(t) = \frac{V_m}{T}t \quad 0 \le t < T$$

$$a_o = \frac{1}{T}\int_0^T v(t)\,dt = \frac{1}{T}\int_0^T \left(\frac{V_m}{T}t\right) dt = \frac{V_m}{2}$$ (Average value.)

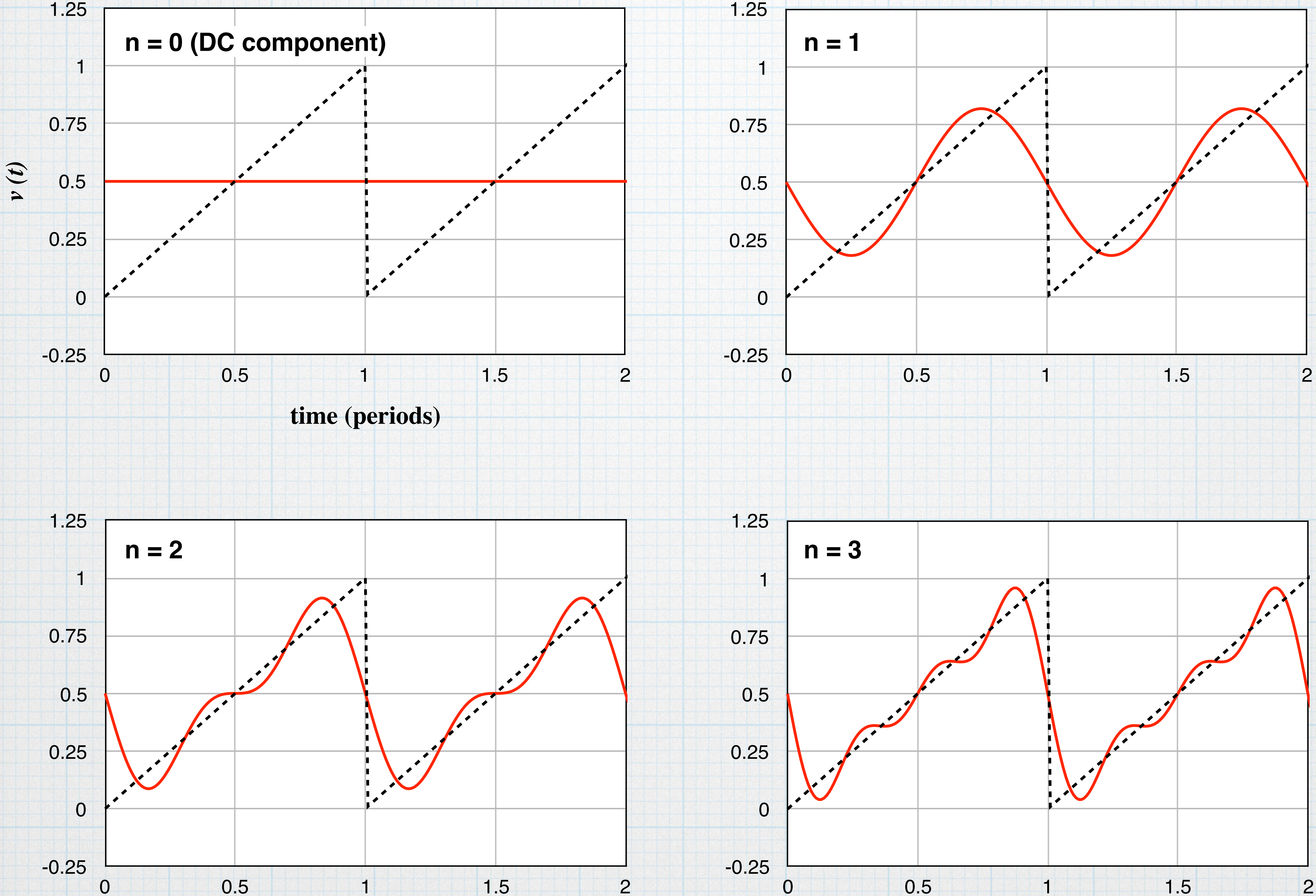
$$a_n = \frac{2}{T}\int_0^T v(t)\cos\left(n\omega_o t\right) dt = \frac{2}{T}\int_0^T \left(\frac{V_m}{T}t\right) \cos\left(n\omega_o t\right) dt = 0 \;\; !!!$$

$$b_n = \frac{2}{T}\int_0^T v(t)\sin\left(n\omega_o t\right) dt = \frac{2}{T}\int_0^T \left(\frac{V_m}{T}t\right) \sin\left(n\omega_o t\right) dt = -\frac{V_m}{n\pi}$$

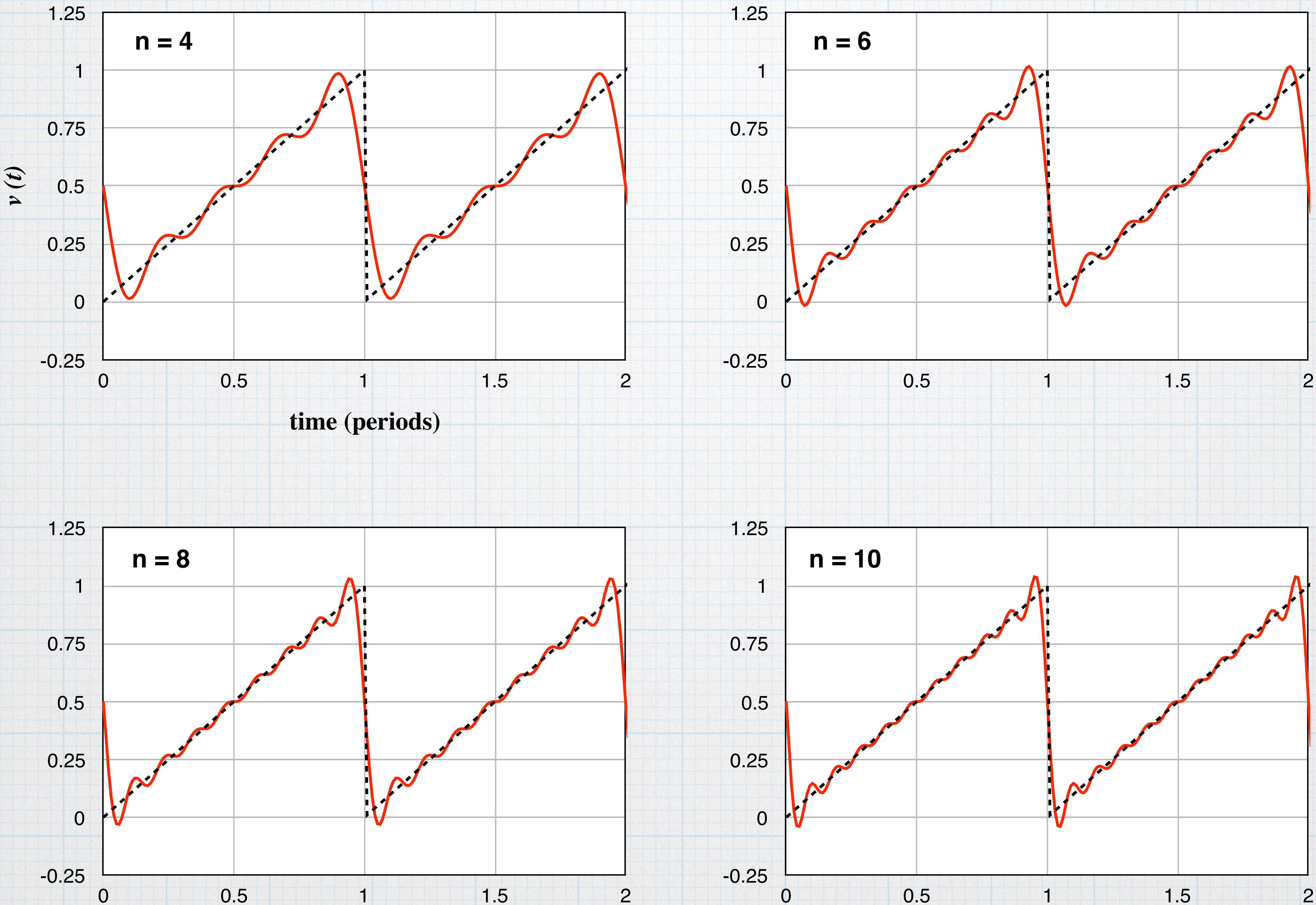
$$v(t) = \frac{V_m}{2} - \frac{V_m}{\pi}\sum_{n=1}^{\infty} \frac{\sin\left(n\omega_o t\right)}{n}$$

Work out these integrals for yourself.

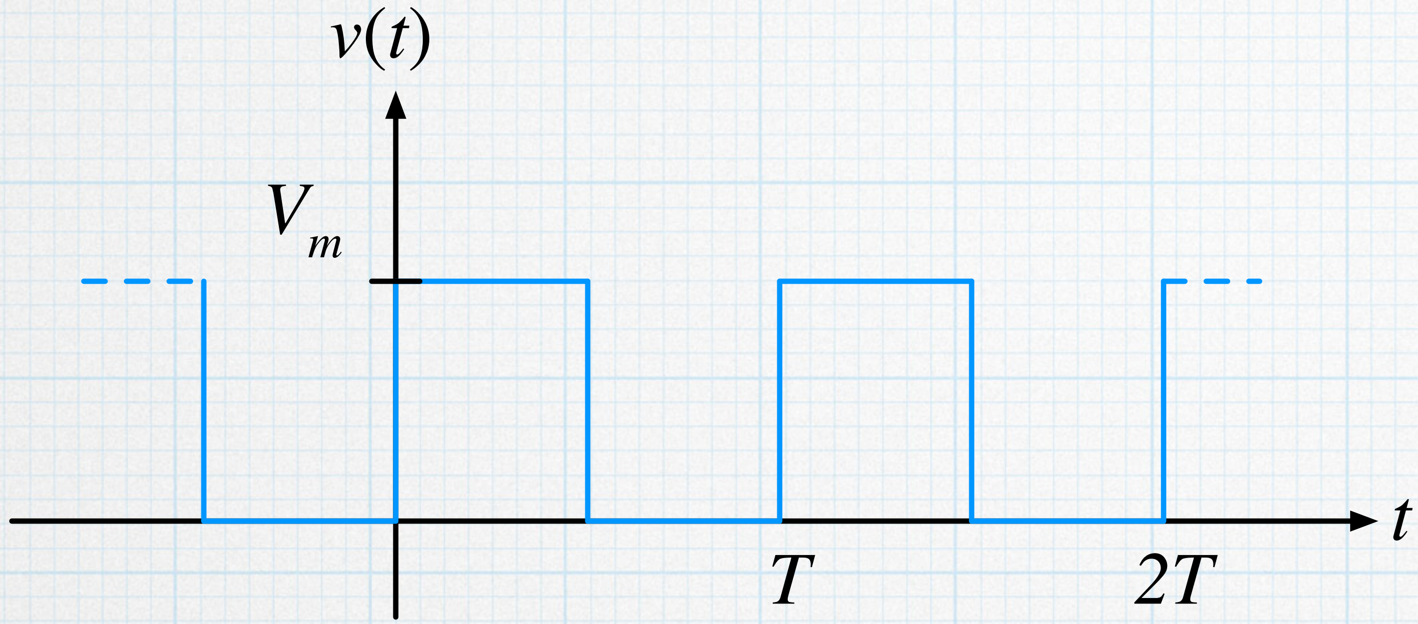
Sawtooth


n = 0 (DC component)
n = 1
n = 2
n = 3
v (t)
time (periods)
1.25
1
0.75
0.5
0.25
0
-0.25
0
0.5
1
1.5
2

Sawtooth



Example - square (pulse)



$v(t) = V_m$ for $0 \le t < T/2$

$= 0$ for $T/2 \le t < T$

(50% duty cycle)

$$a_o = \frac{1}{T}\int_0^{T/2} V_m \, dt = \frac{V_m}{2}$$ (Average value.)

$$a_n = \frac{2}{T}\int_0^{T/2} V_m \cos\left(n\omega_o t\right) dt = 0$$ (Again.)

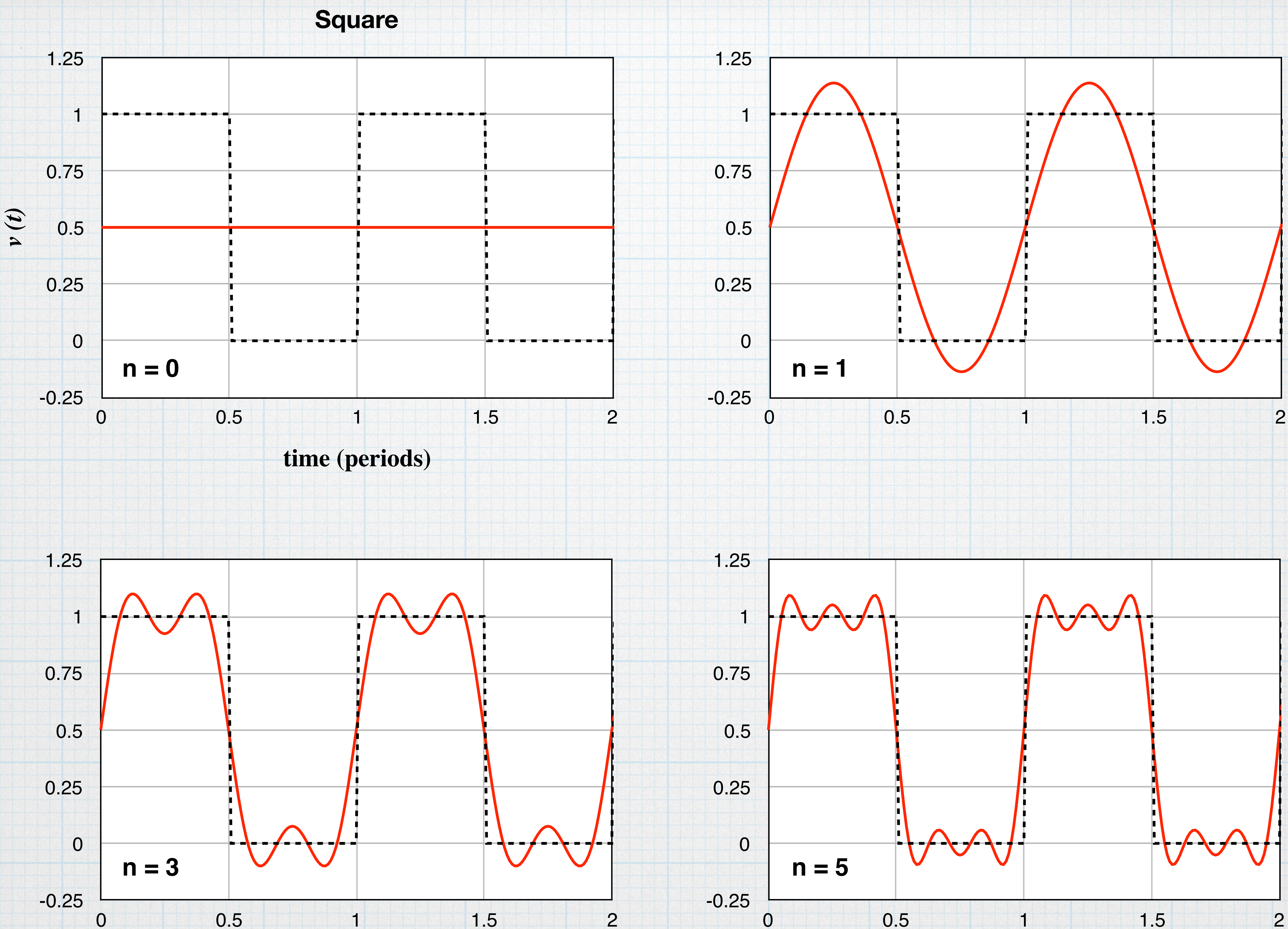
$$b_n = \frac{2}{T}\int_0^{T/2} V_m \sin\left(n\omega_o t\right) dt = \frac{2V_m}{n\pi}$$ if n is odd.

$= 0$ if n is even.

Only the odd terms survive, $n = 1, 3, 5, \ldots$

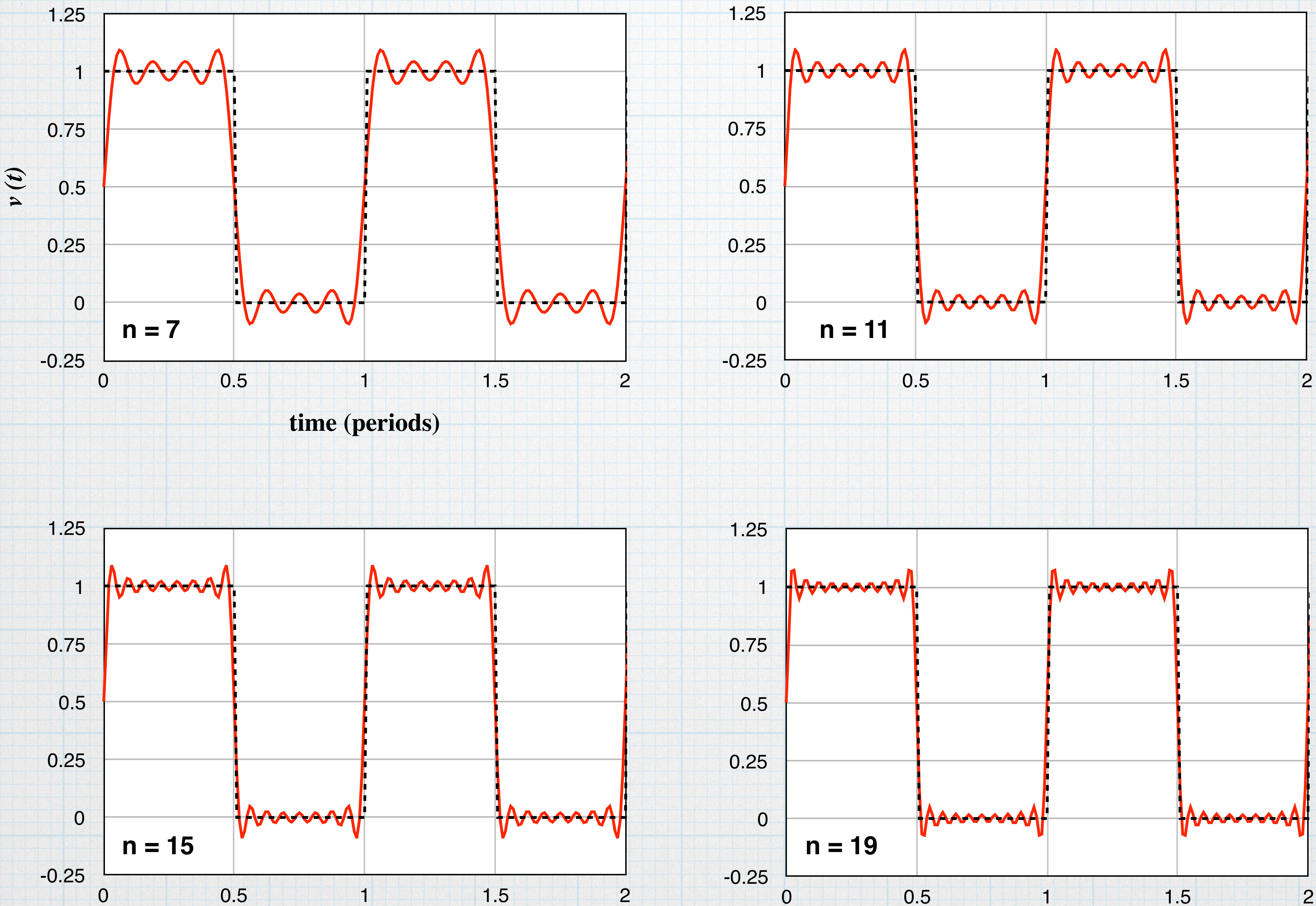
$$v(t) = \frac{V_m}{2} + \frac{2V_m}{\pi}\sum_{n \text{ odd}} \frac{\sin\left(n\omega_o t\right)}{n}$$

Work out the integrals for yourself.

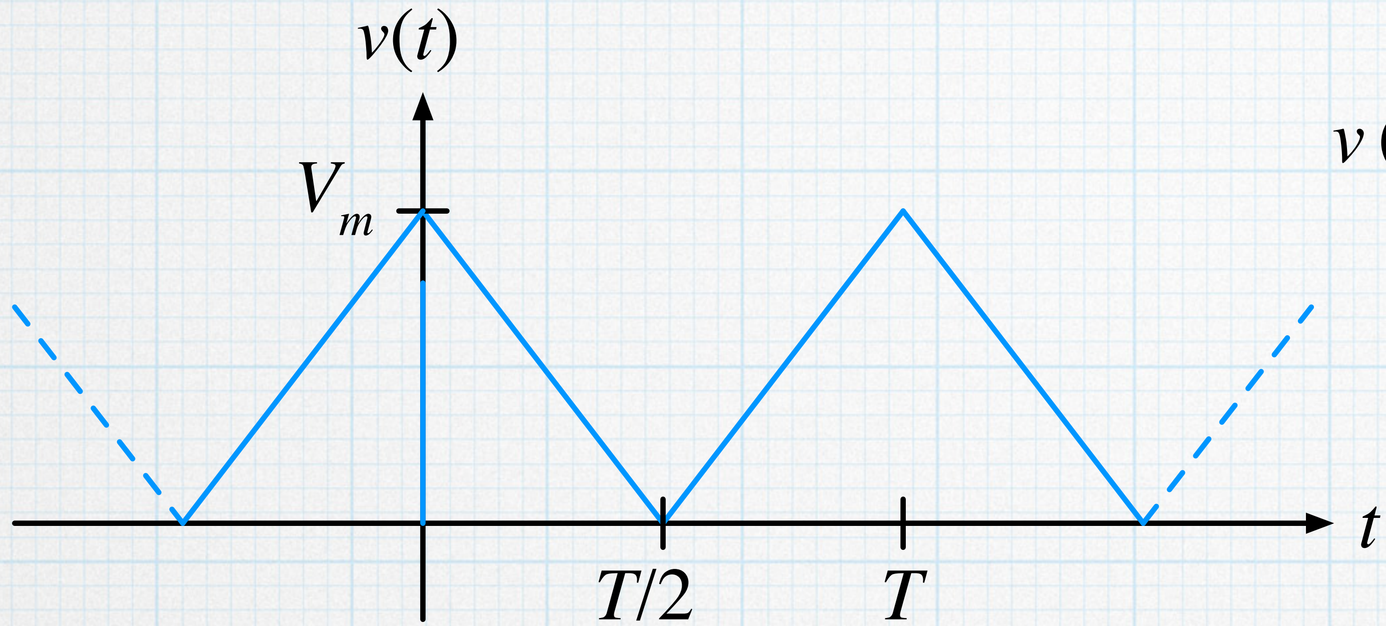


Square
v (t)
time (periods)
n = 0
n = 1
n = 3
n = 5
1.25
1
0.75
0.5
0.25
0
-0.25
0
0.5
1
1.5
2

Square



Example - triangle



$$v(t) = \begin{cases} \frac{V_m}{T}(T - 2t) & 0 \leq t \leq T/2 \\ \frac{V_m}{T}(2t - T) & T/2 \leq t \leq T \end{cases}$$

$$a_o = \frac{1}{T}\int_0^{T/2} \frac{V_m}{T}(T - 2t)\,dt + \frac{1}{T}\int_{T/2}^{T} \frac{V_m}{T}(2t - T)\,dt = \frac{V_m}{2}$$

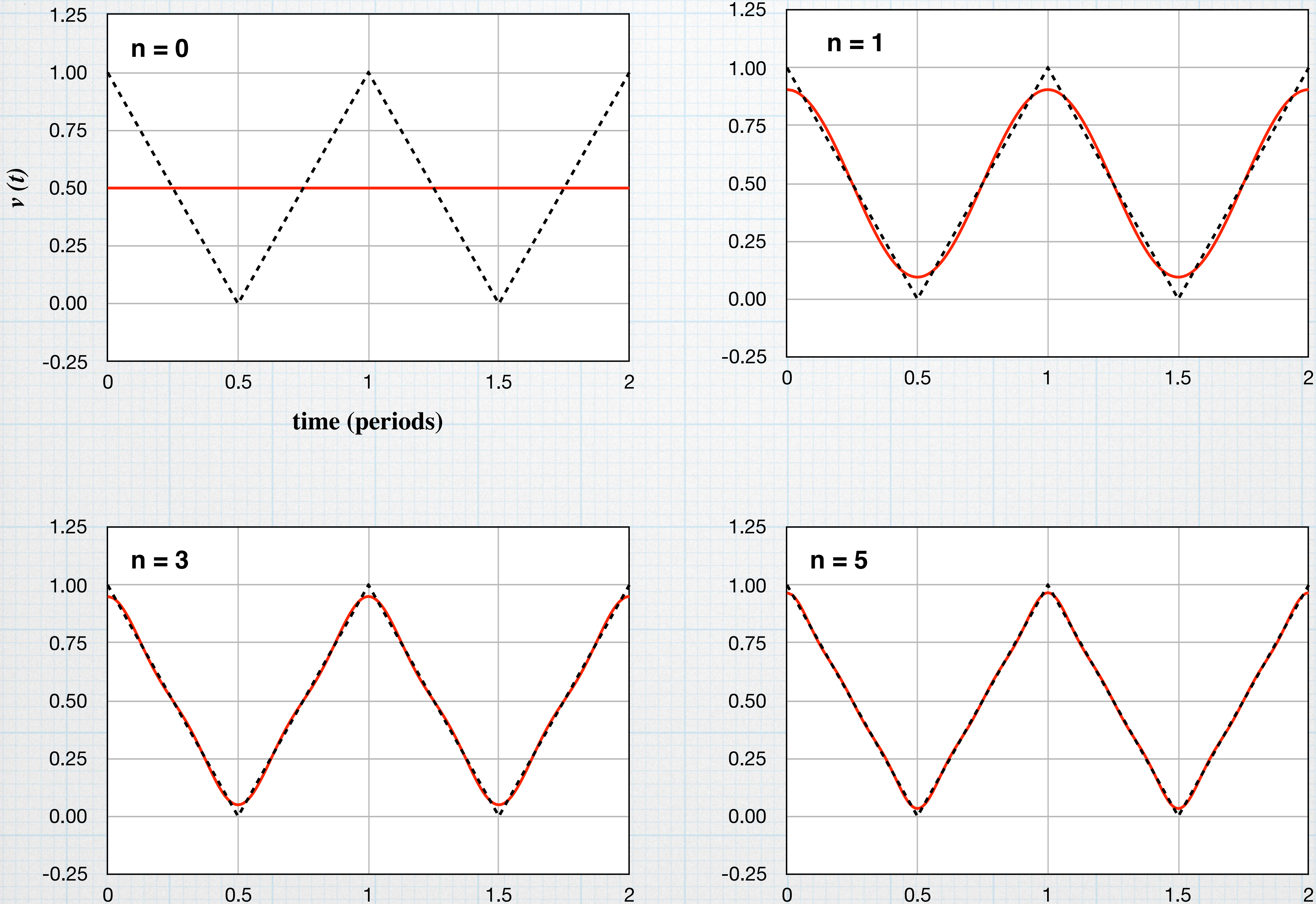
$$a_n = \frac{2}{T}\int_0^{T/2} \frac{V_m}{T}(T - 2t)\cos\left(n\omega_o t\right)dt + \frac{2}{T}\int_{T/2}^{T} \frac{V_m}{T}(2t - T)\cos\left(n\omega_o t\right)dt$$

$$= \begin{cases} \frac{4V_m}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

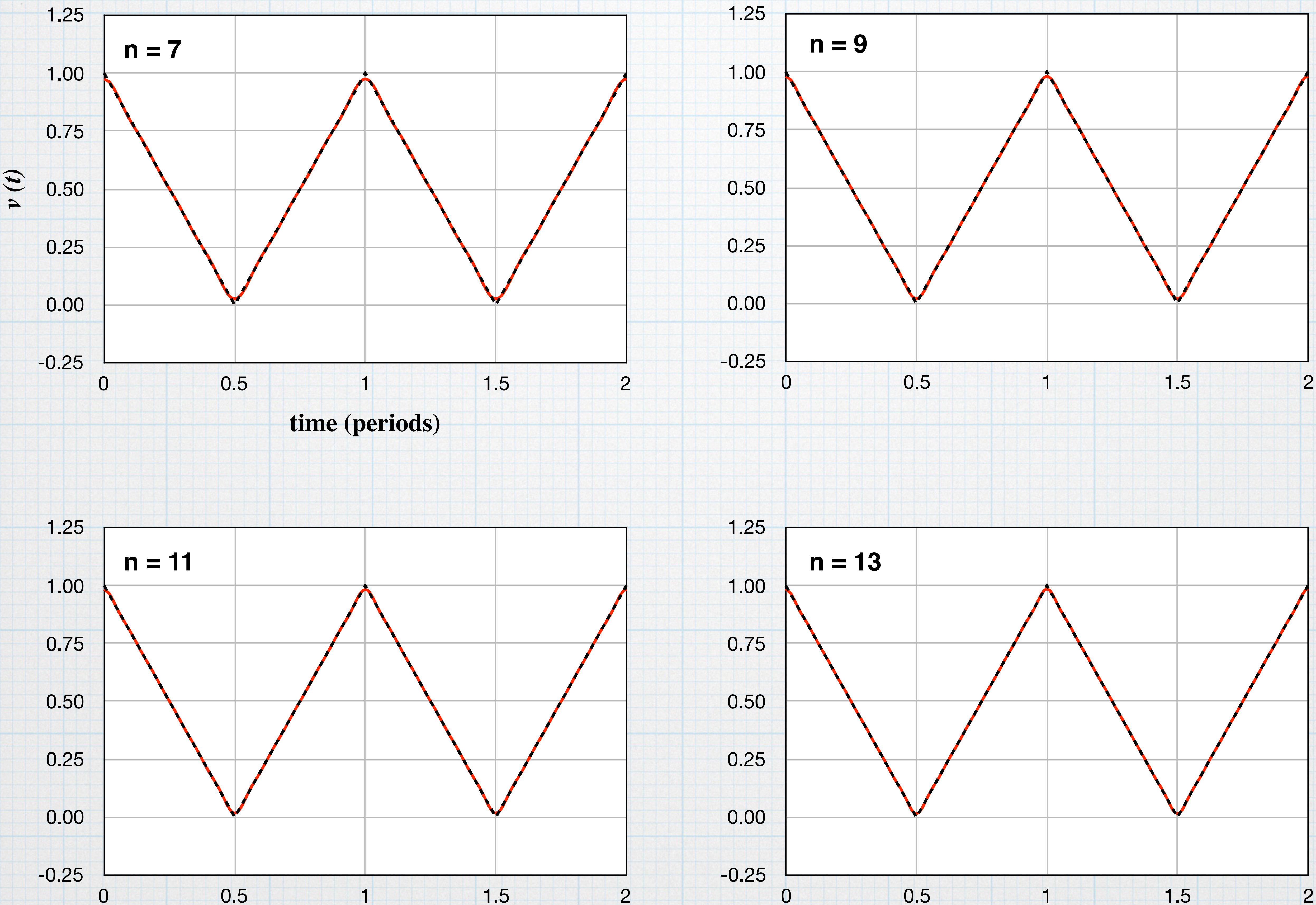
$$v(t) = \frac{V_m}{2} + \frac{4V_m}{\pi^2}\sum_{n \text{ odd}} \frac{\cos\left(n\omega_o t\right)}{n^2}$$

$b_n = 0$ (By symmetry.)

Triangle



Triangle


n = 7
n = 9
n = 11
n = 13
v (t)
time (periods)
1.25
1.00
0.75
0.50
0.25
0.00
-0.25
0
0.5
1
1.5
2

Alternative trig version

$$v(t) = a_o + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t) \right]$$

In general, at each harmonic, there is a pair of terms:

$$a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)$$

Recall the trig identity from our days with AC analysis in 201

$$A \cos x + B \sin x = C \cos(x - \delta x) \qquad C = \sqrt{A^2 + B^2} \qquad \delta x = \arctan\left(\frac{B}{A}\right)$$

We can combine the two terms at $n\omega_o$ into one:

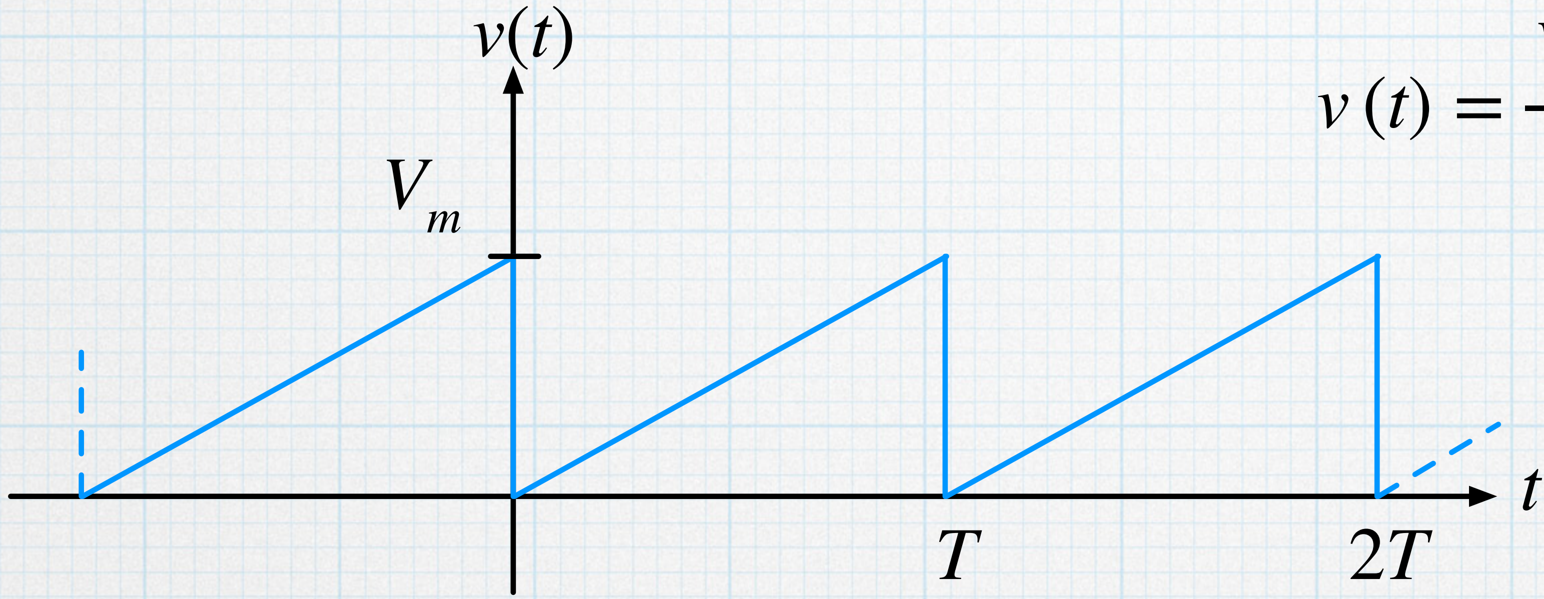
$$a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t) = c_n \cos(n\omega_o t + \theta_n)$$

$$v(t) = a_o + \sum_{n=1}^{\infty} c_n \cos(n\omega_o t + \theta_n)$$

Going a step further, we recognize that the DC term can be viewed as a cosine with zero frequency, i.e. $a_o = a_o \cos(n\omega_0 t)$ with $n = 0$.

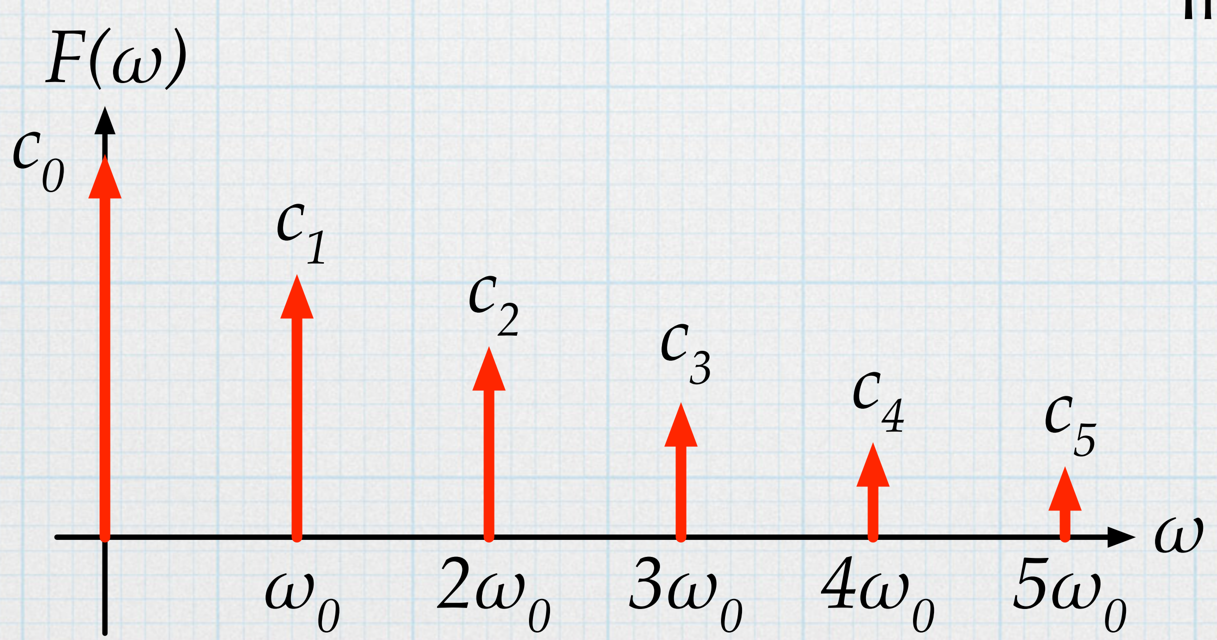
$$v(t) = \sum_{n=0}^{\infty} c_n \cos(n\omega_o t + \theta_n)$$

The reduced form doesn't really offer a computational advantage over using separate sines and cosines. It is still necessary to calculate to parameters at each frequency. It does allow us to view a signal in a slightly different fashion – as a line spectrum.

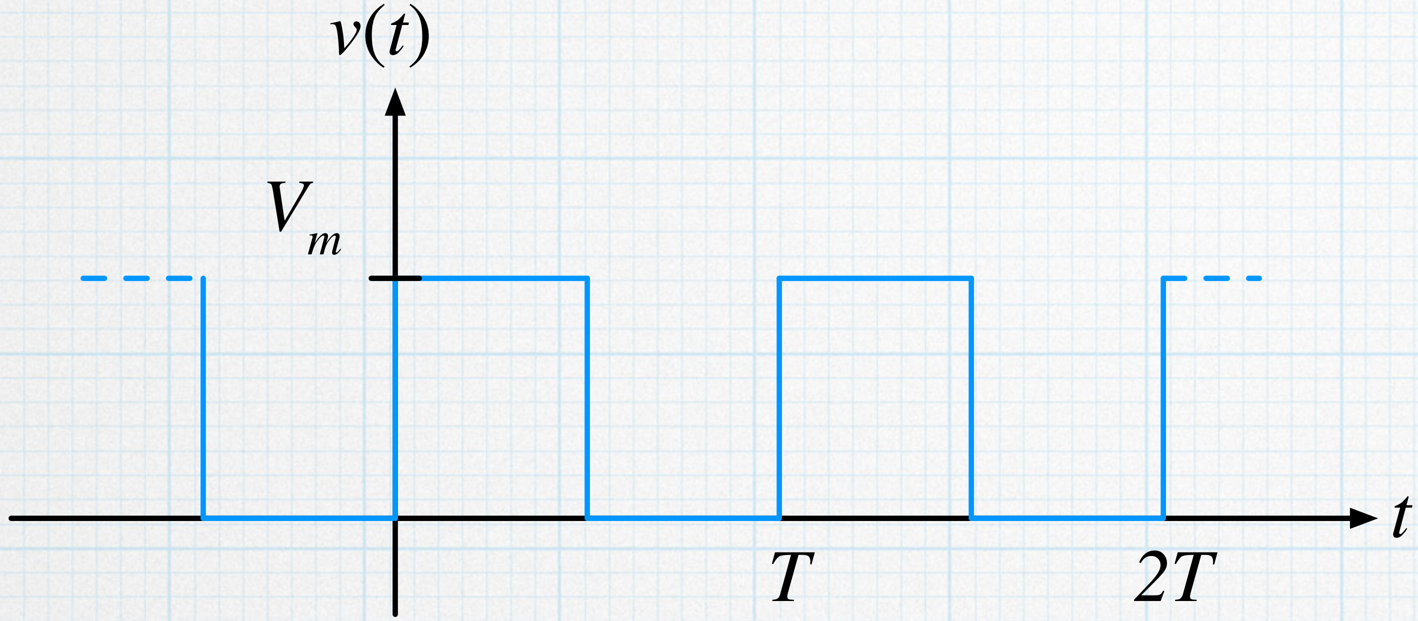


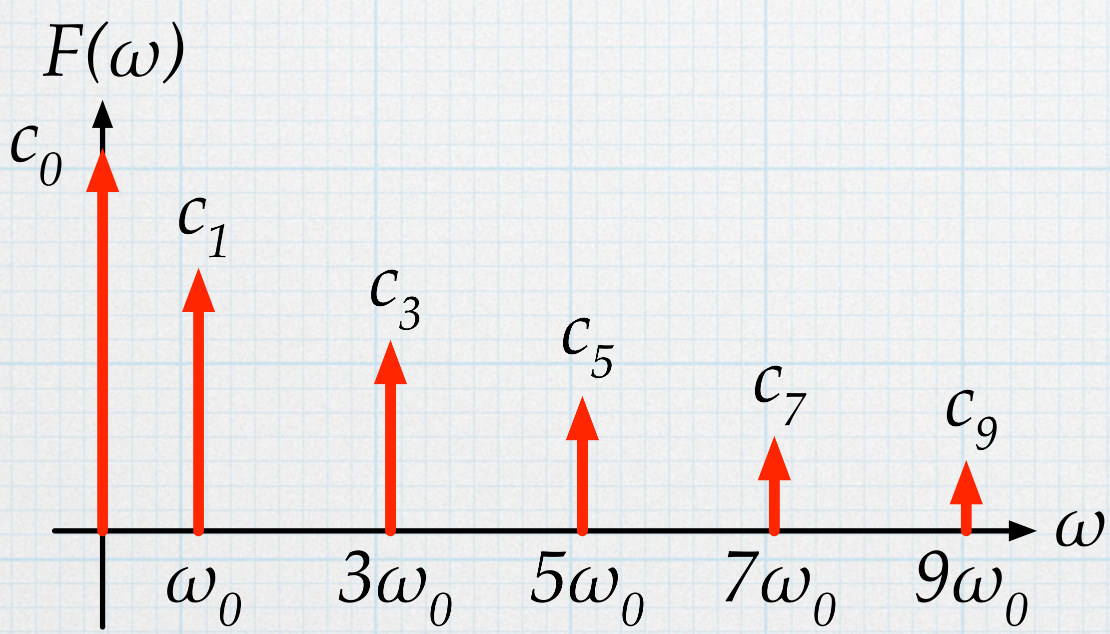
$$v(t) = \frac{V_m}{2} - \frac{V_m}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\omega_o t)}{n}$$

In this case, $c_n = b_n$ and $\theta_n = 90°$.



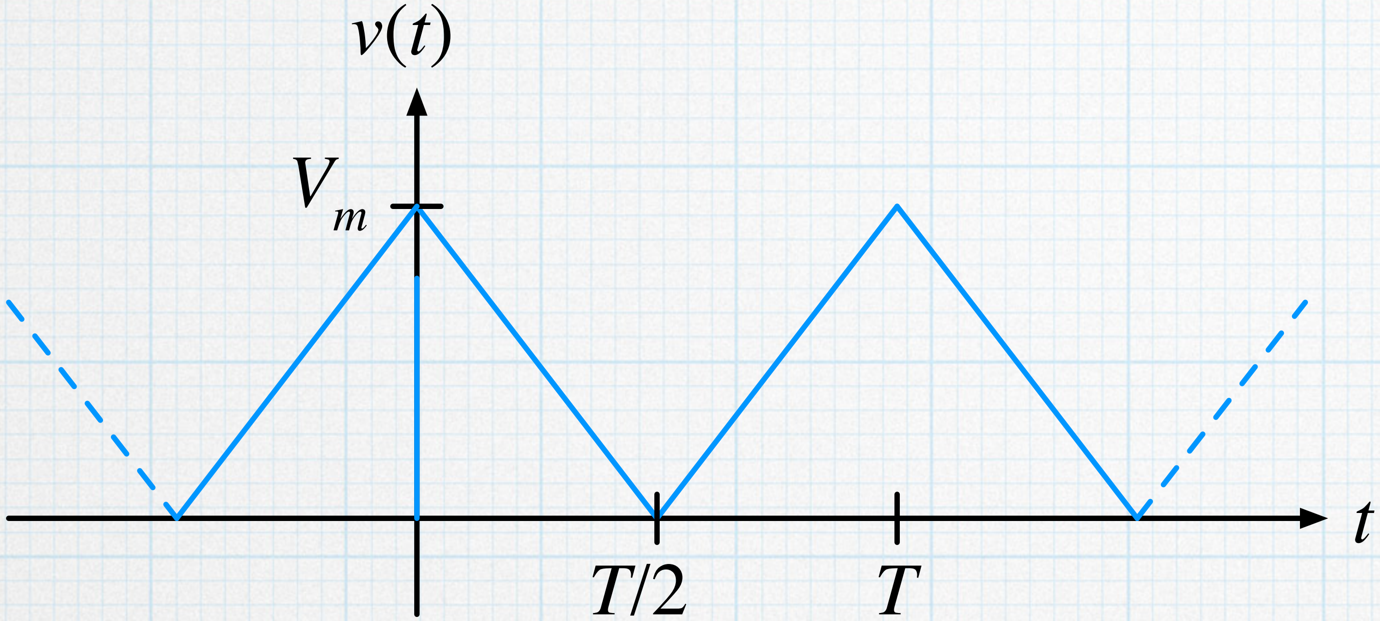
The Fourier series is a type of transform, and we can view the function in "Fourier-series" space.



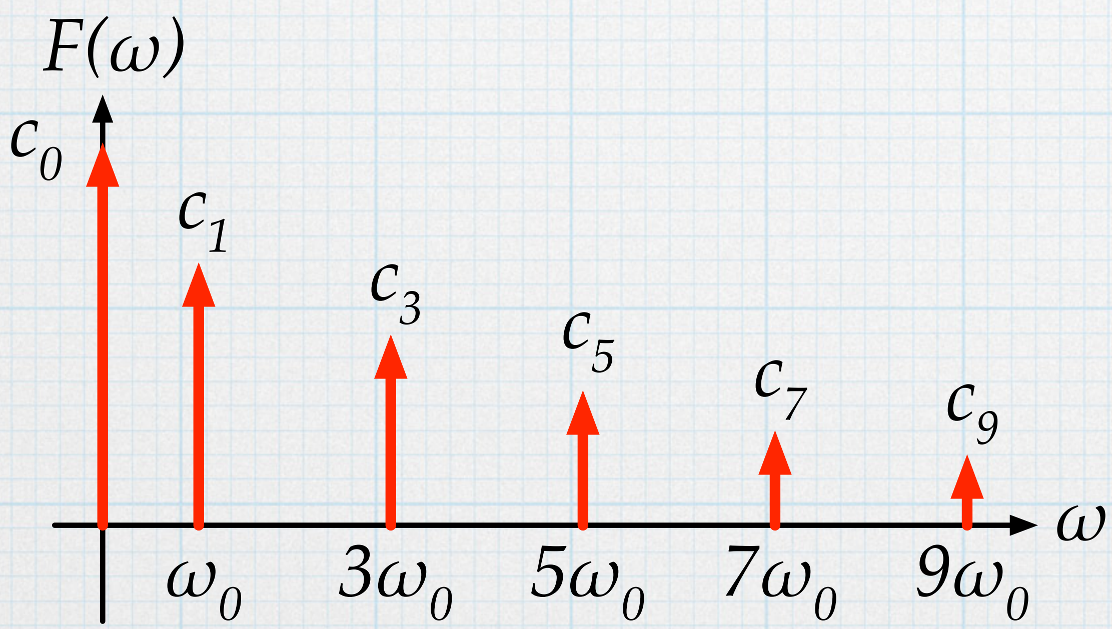


$$v(t) = \frac{V_m}{2} + \frac{2V_m}{\pi} \sum_{n\ \text{odd}} \frac{\sin\left(n\omega_o t\right)}{n}$$

Again, $c_n = b_n$ and $\theta_n = 90°$.

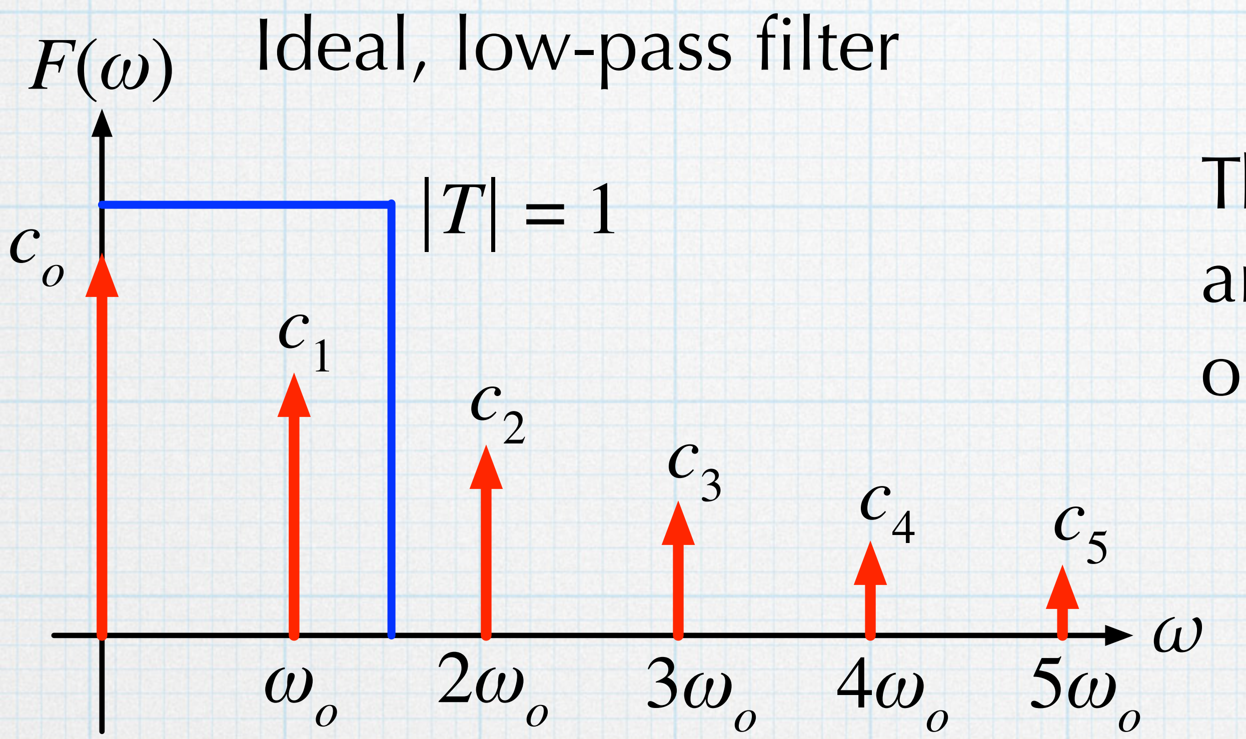


$$v(t) = \frac{V_m}{2} + \frac{4V_m}{\pi^2} \sum_{n\ \mathrm{odd}} \frac{\cos\left(n\omega_o t\right)}{n^2}$$



Now, $c_n = a_n$ and $\theta_n = 0$.

What happens when we apply a sinusoidal signal to a filter. The Fourier series line spectrum shows us exactly what will happen. Consider a low-pass filter. (Make it ideal, just for simplicity.)

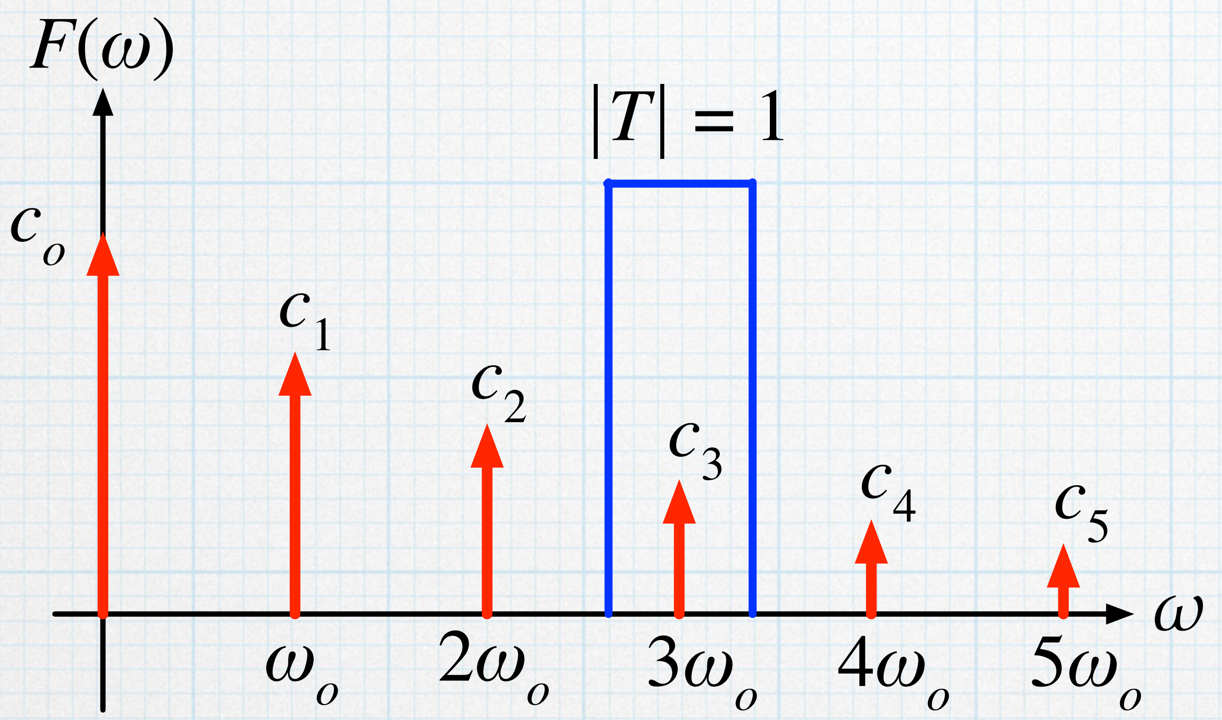


The signal components at c_o and c_1 pass through. All higher order components are cut-off.

$$v_o(t) = c_o + c_1 \cos\left(n\omega_o t + \theta_1\right)$$

A DC and single single frequency of the sine wave come out!

Or an ideal band-pass filter centered at $3\omega_o$:



$$v_o(t) = c_3 \cos\left(3\omega_o t + \theta_3\right)$$