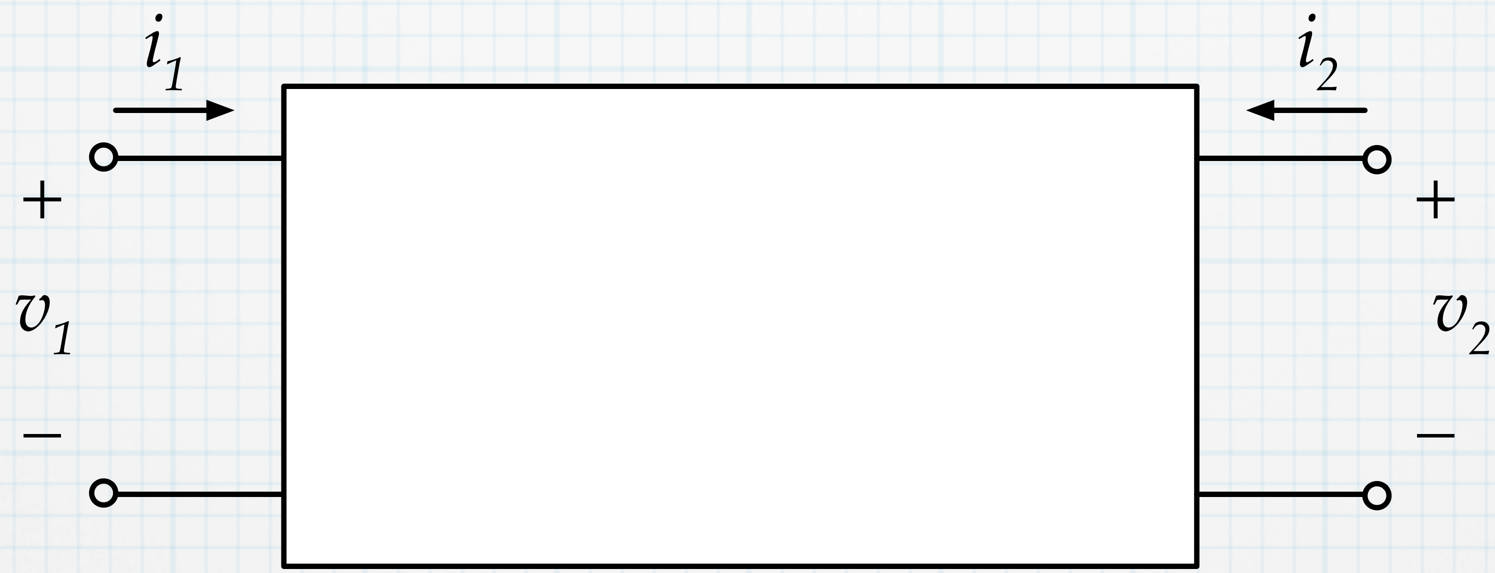
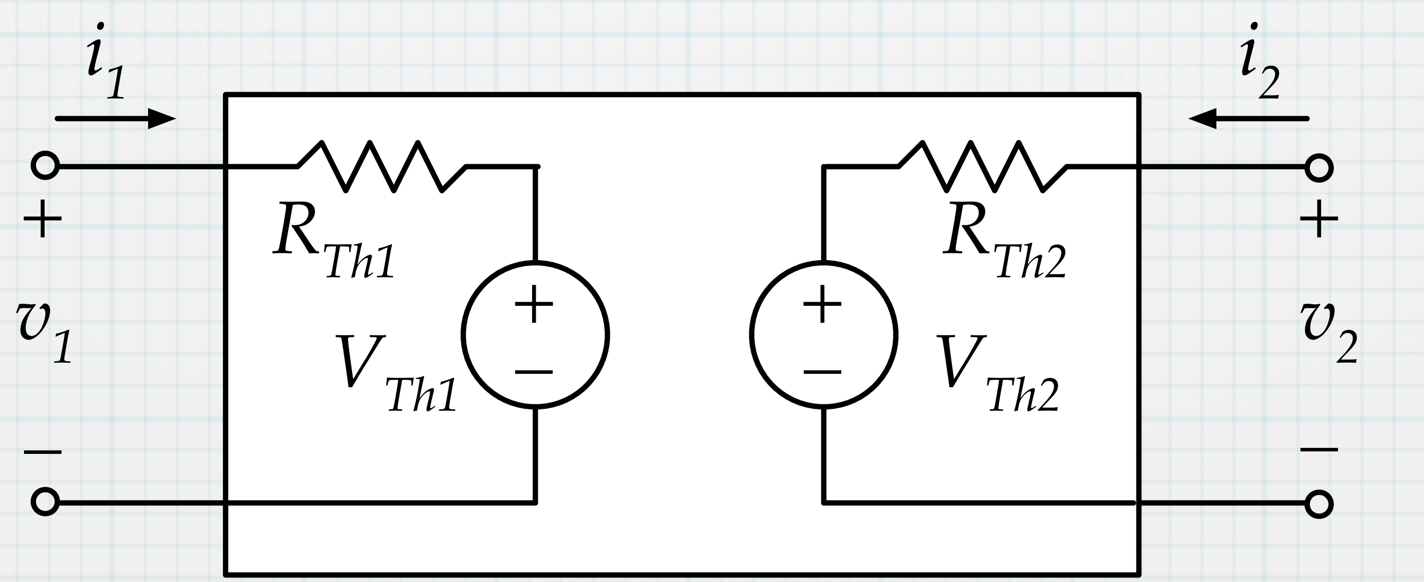
Two-port circuits

With the idea of the Thevenin (or Norton) equivalent, we saw that we could represent the behavior of a circuit at a port (pair of nodes) using a simple source-resistor combination.

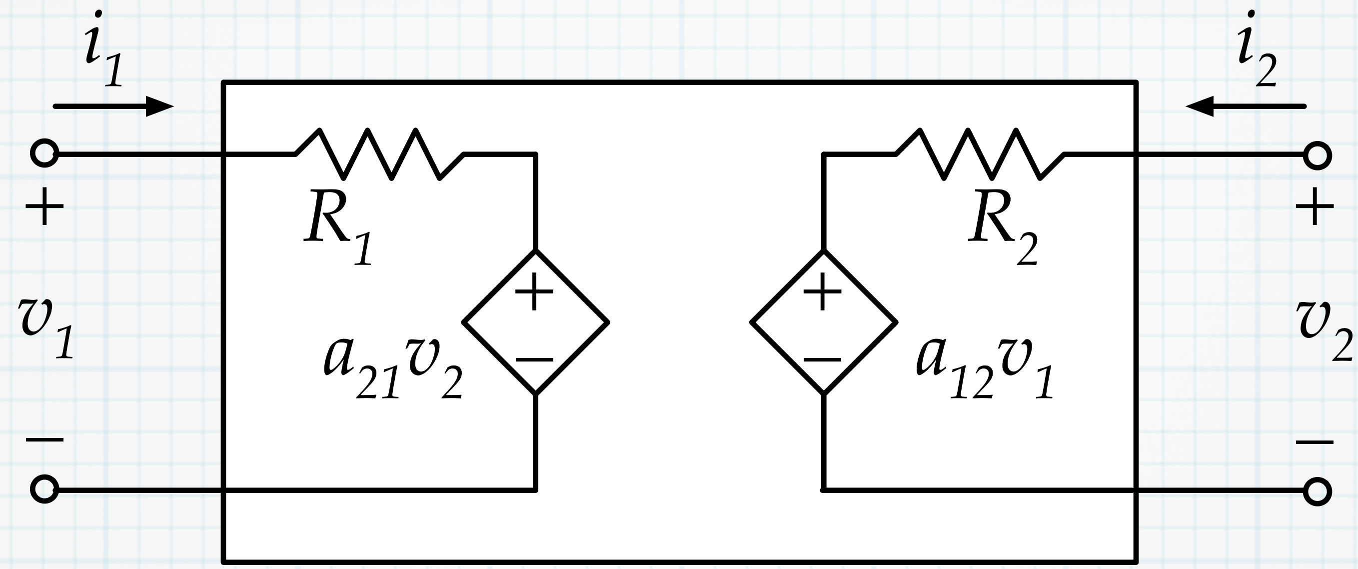
What if we have two ports? A circuit that has an input and an output would need two ports – for example, an amplifier.



Put a Thevenin at each port.

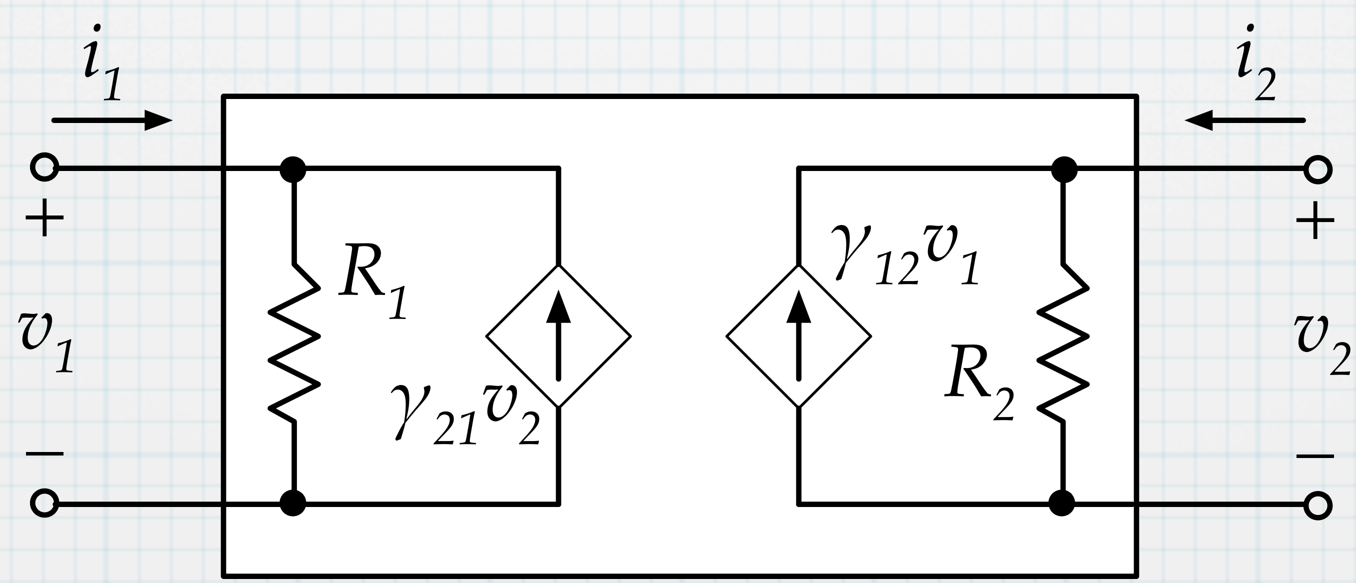


Not really correct – just two one ports.

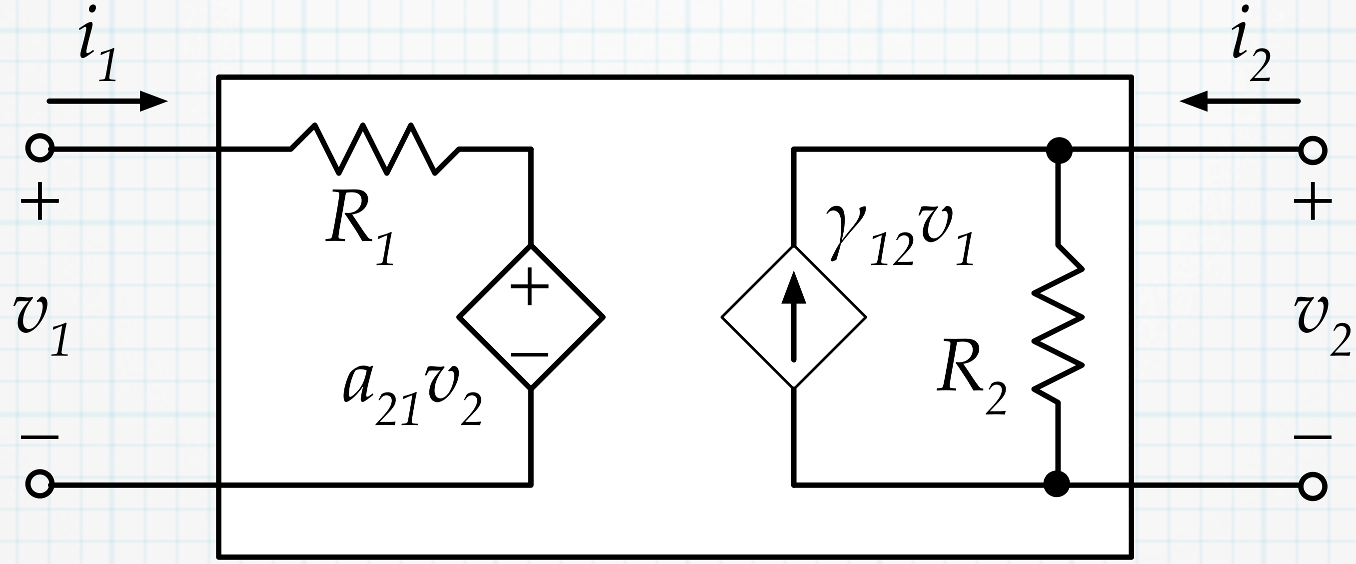


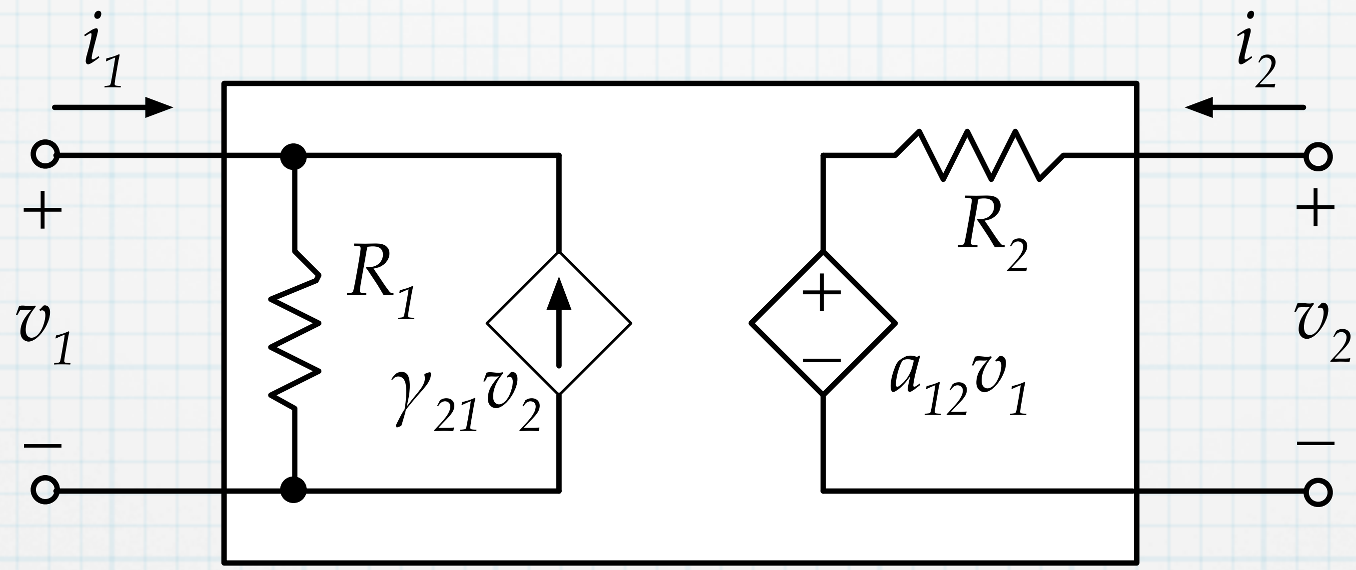
Need to use dependent sources. Now port 1 is connected to port 2 (the voltage at port 2 affects port 1) and vice-versa. This is the two-port version of the Thevenin equivalent idea. We can use this to model circuits that have an input and output.

If desired, we can do a source transformation at each end.



Or mix and match...



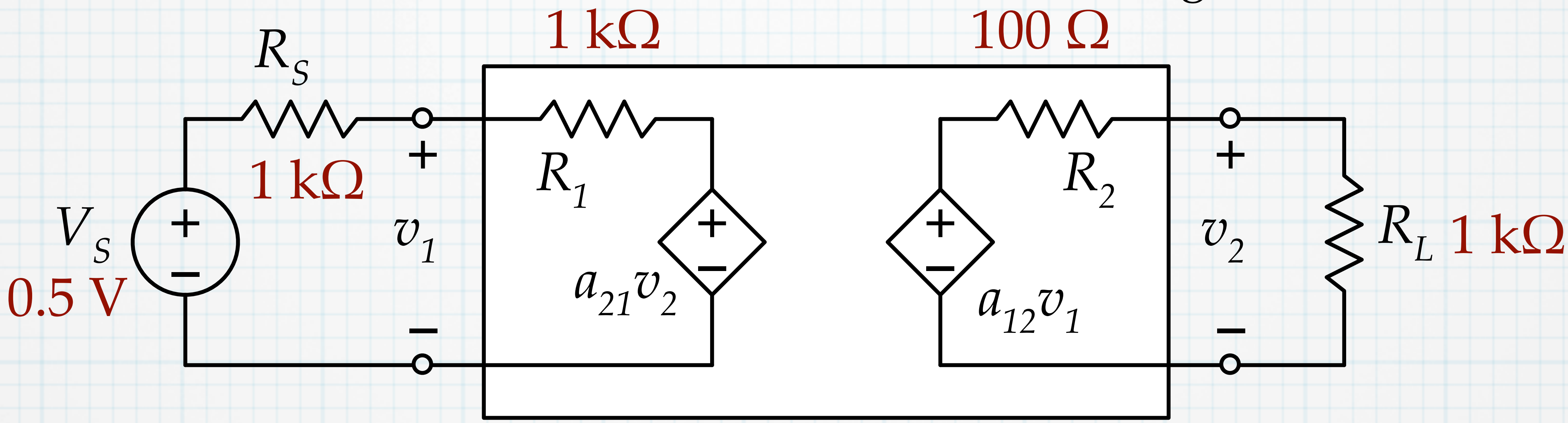


Use whatever combination is most advantageous in doing the circuit analysis.

The general convention is define the current as inward at each port, even if we know that it is flowing out.

An example

Consider a two port circuit as shown below, with a voltage source attached to the left and a load resistor attached to the right.



$a_{21} = 0.1$ $\qquad$ $a_{12} = 10$

$$v_2 = \frac{R_L}{R_L + R_2} a_{12} v_1 \longrightarrow v_1 = \frac{R_L + R_2}{R_L} \frac{v_2}{a_{12}}$$

$$v_1 = i_1 R_1 + a_{21} v_2 = \left(\frac{V_S - a_{21} v_2}{R_1 + R_S} \right) R_1 + a_{21} v_2$$

$$\frac{R_L + R_2}{R_L} \frac{v_2}{a_{12}} = \left(\frac{V_S - a_{21} v_2}{R_1 + R_S} \right) R_1 + a_{21} v_2$$

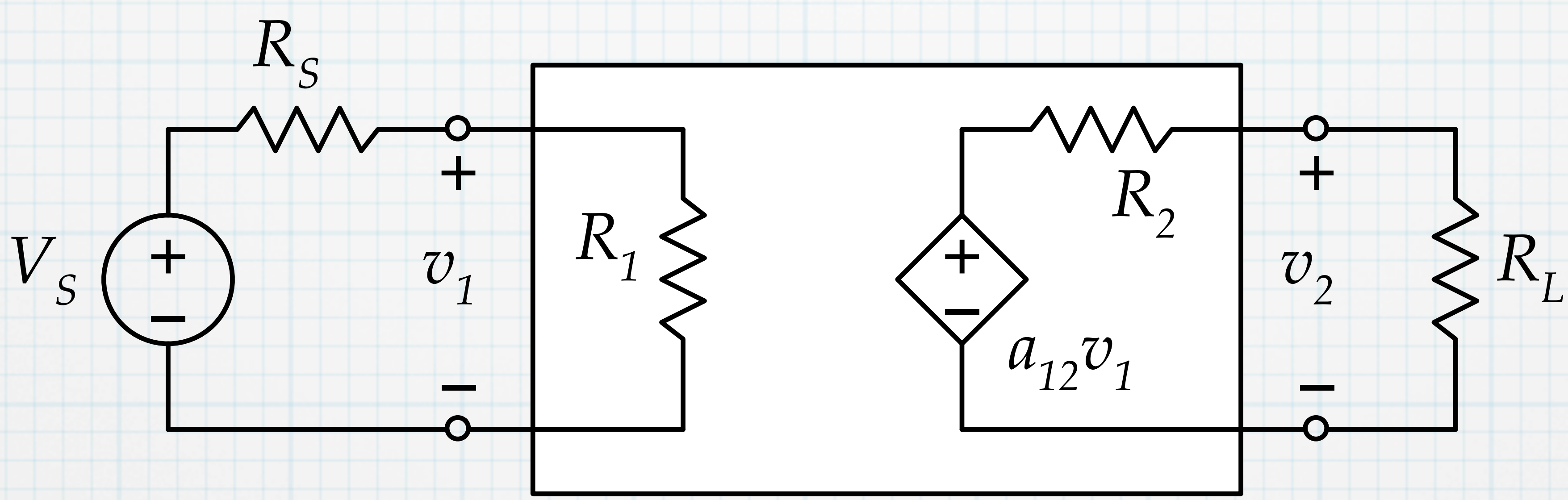
Solve for v_2.

$$v_2 = \frac{R_1 R_L a_{12}}{(R_1 + R_S)(R_L + R_2) + a_{12} a_{21} \left[R_1 R_L - R_L (R_1 + R_S) \right]} V_S = 4.167 \text{ V}$$

$$v_2 = \frac{R_1 R_L a_{12}}{(R_1 + R_S)(R_L + R_2) + a_{12} a_{21} \left[R_1 R_L - R_L (R_1 + R_S)\right]} V_S$$

If $a_{21} = 0$

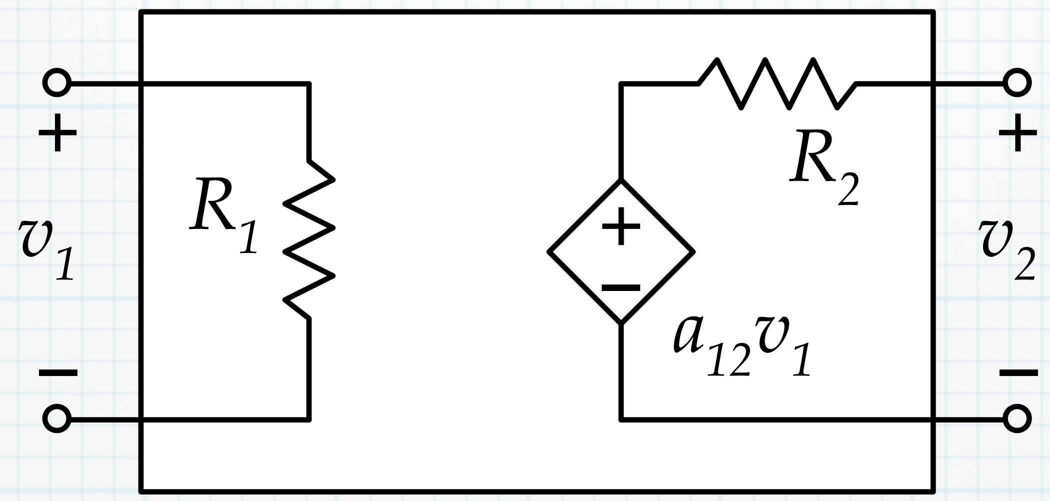
$$v_2 = \frac{R_1 R_L a_{12}}{(R_1 + R_S)(R_L + R_2)} V_S = 4.545 \text{ V}$$



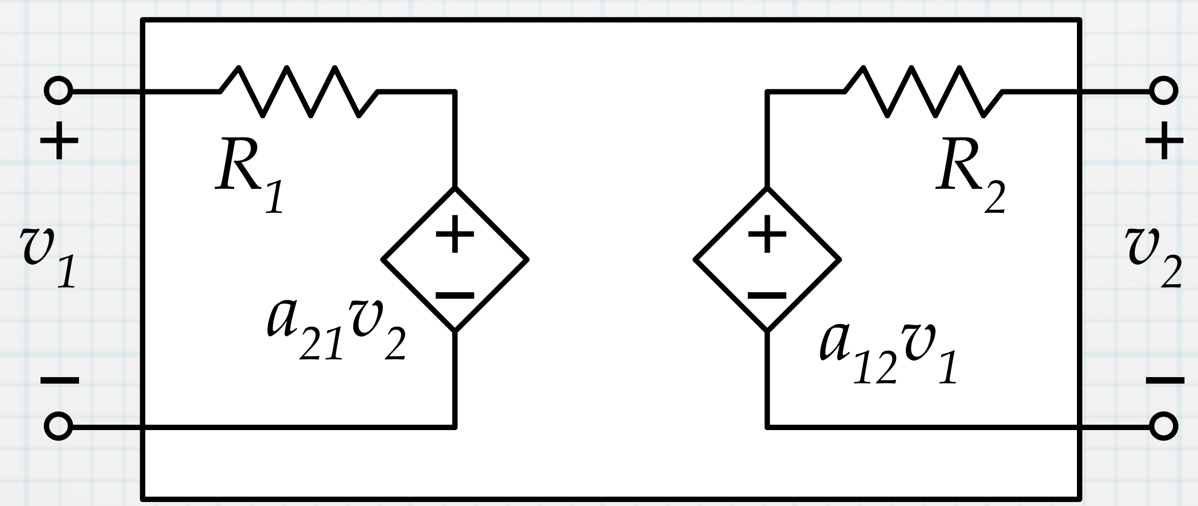
$$v_1 = \frac{R_1}{R_1 + R_S} V_S \qquad v_2 = \frac{R_L}{R_L + R_2} a_{12} v_1$$

$$v_2 = \left(\frac{R_S}{R_1 + R_S}\right) a_{12} \left(\frac{R_L}{R_2 + R_L}\right) V_S$$

For amplifiers, the simpler model with $a_{21} = 0$ is generally sufficient to provide a good description of the circuit's behavior.



In general, we need to include all four components if we want a complete description of an unknown circuit.



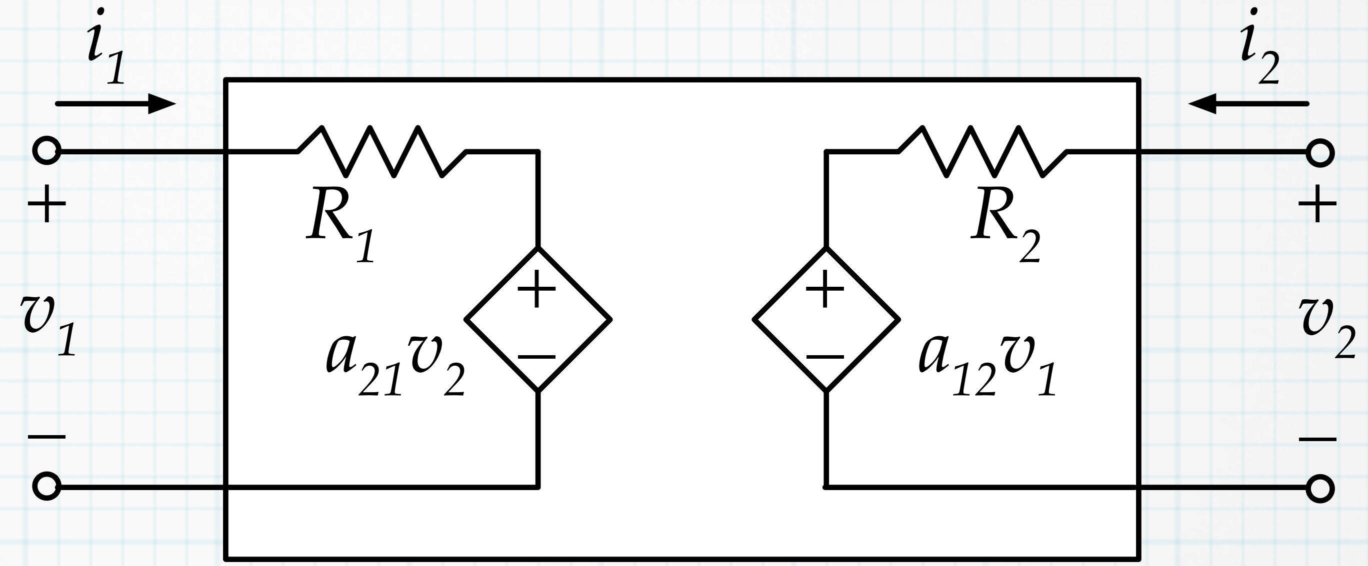
Determining the parameters for a two-port

Similar to finding the parameters for a one-port Thevenin. Since there are four parameters, we expect four measurements or calculations.

But first, note that since there is no "internal" independent source, we need to provide an independent source at one side so that we can get a response at the other side. We can call this a "test source". It can be either type (voltage or current) and it can have any value that we like. Typically, we might choose 1 V or 1 A, or we might leave it in symbolic form. It doesn't really matter because the response will always be proportional to the value of the test source.

Like the Thevenin approach, we might expect to look for open-circuit voltage and short-circuit currents at each side to determine the voltages and resistances.

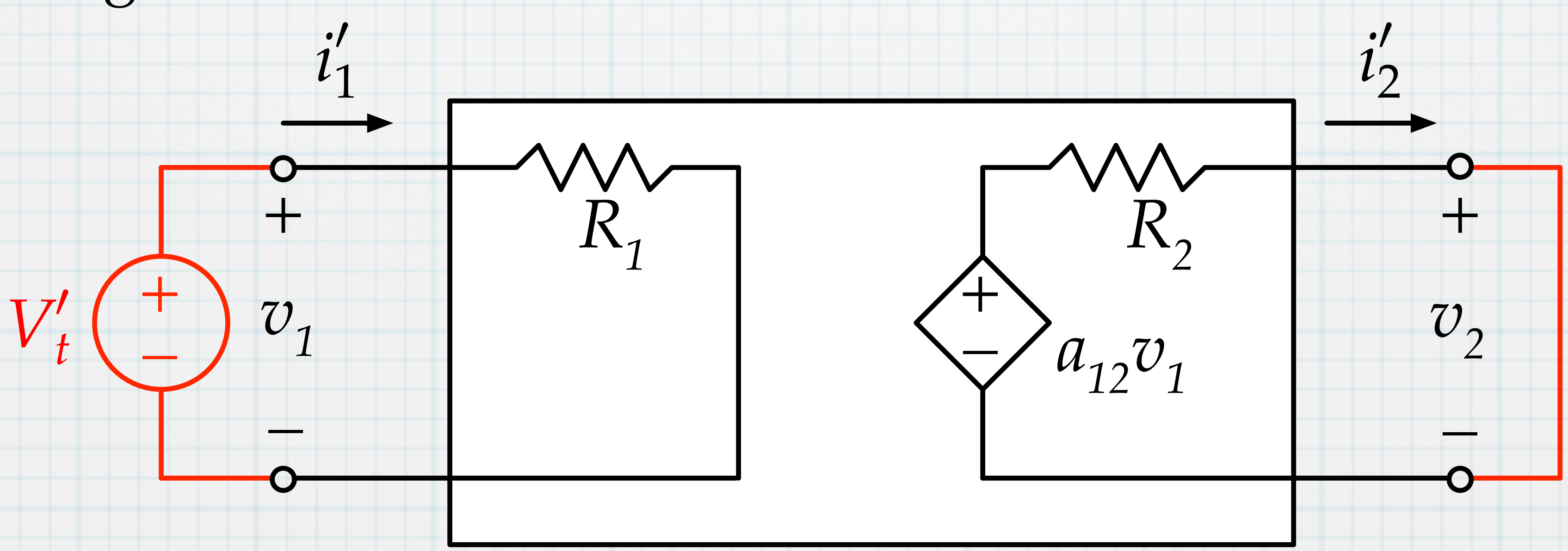
However, it turns out that the open-circuit measurements are not really necessary.



1. Apply a test voltage, V_t' at port 1. $v_1 = V_t'$.
2. Short the terminals at port 2, making $v_2 = 0$. This has the effect of making the source $a_{21}v_2 = 0$.
3. Measure, or calculate, the currents at the two ports. (Note that we have changed the direction of i_2.)

$$i_1' = \frac{V_t'}{R_1}$$

$$R_1 = \frac{V_t'}{i_1'}$$

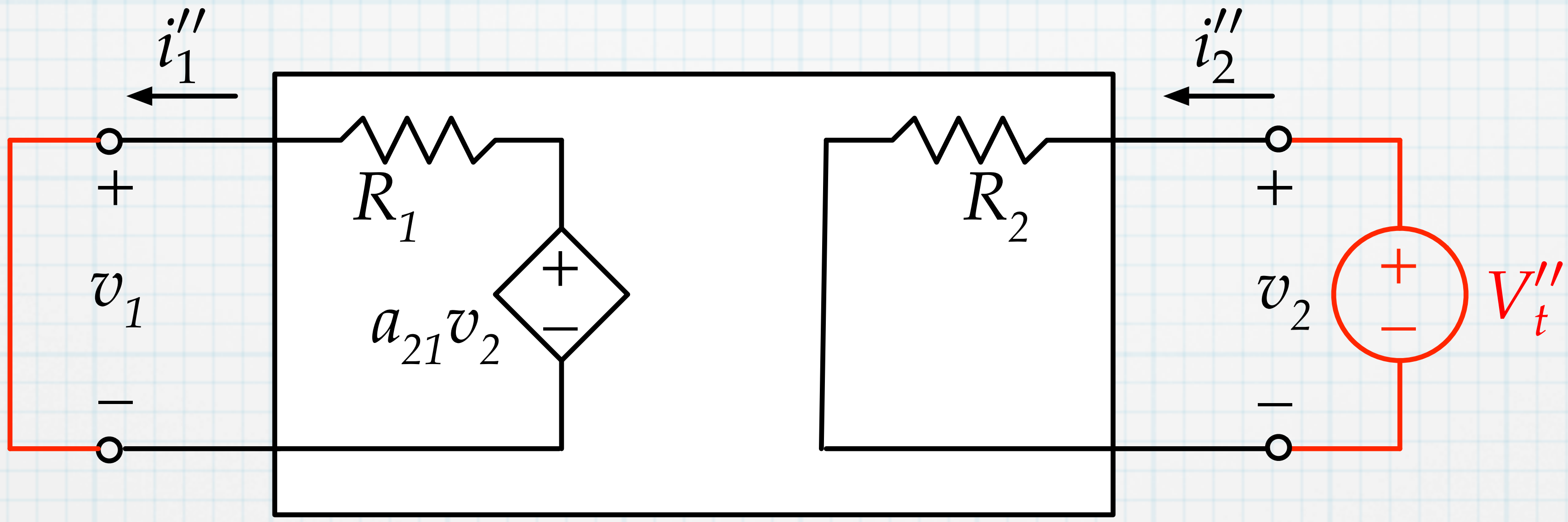


$$i_2' = \frac{a_{12}V_t'}{R_2}$$

$$\frac{R_2}{a_{12}} = \frac{V_t'}{i_2'}$$

Now reverse the experiment:

4. Apply a test voltage, V_t'' at port 2. $v_2 = V_t''$.
5. Short the terminals at port 1, making $v_1 = 0$. This has the effect of making the source $a_{12} v_1 = 0$.
6. Measure, or calculate, the currents at the two ports. (Note that we have changed the direction of i_1.)



$$i_2'' = \frac{V_t''}{R_2}$$

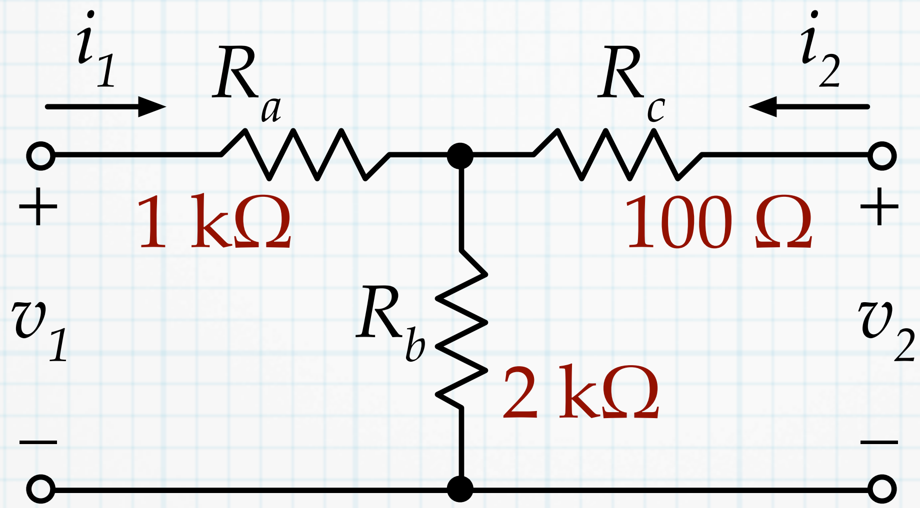
$$R_2 = \frac{V_t''}{i_2''}$$

$$i_1'' = \frac{a_{21} V_t''}{R_1} \qquad \frac{R_1}{a_{21}} = \frac{V_t''}{i_1''}$$

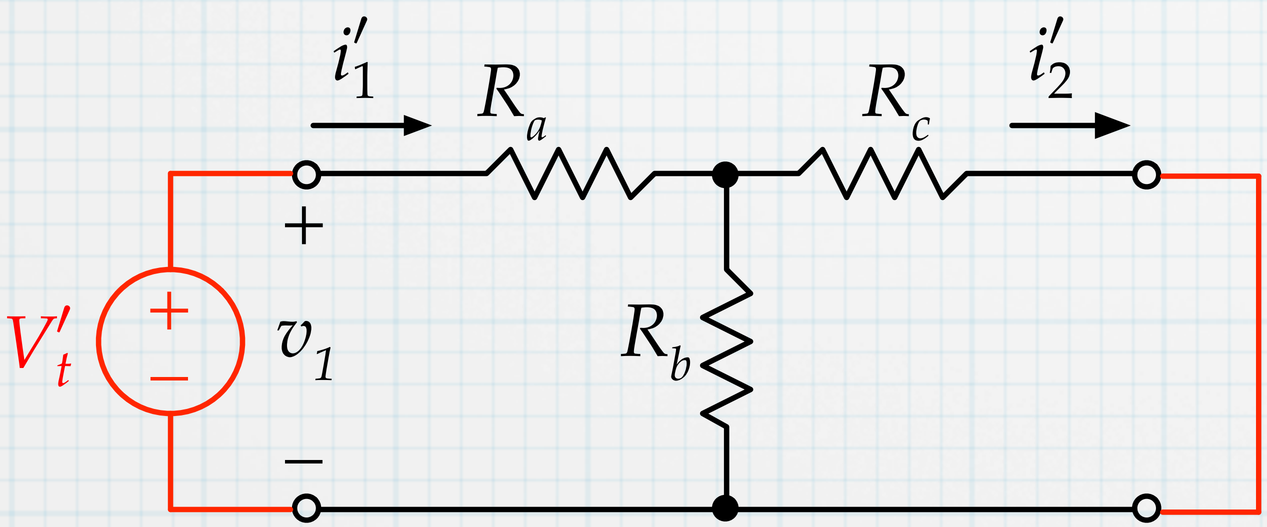
From the measured (calculated) values, we can determine the four parameters for the two-port.

Example

Find the two-port equivalent for the simple T-network shown below.



1. Apply a test voltage at v_1. Short circuit v_2. . Calculate the terminal currents.

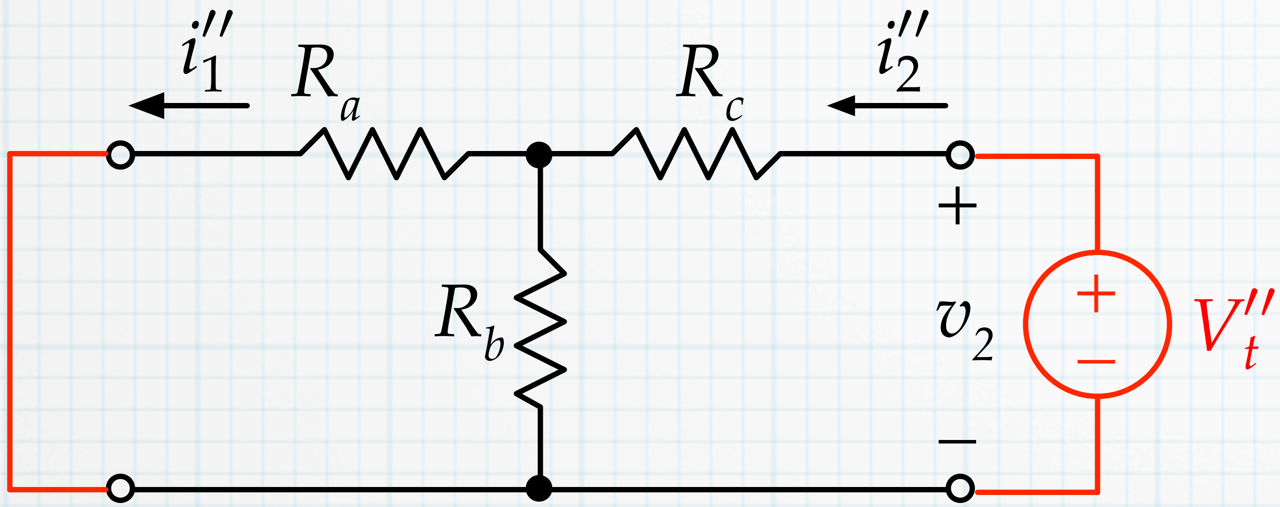


$$i_1' = \frac{V_t'}{R_{eq}} = \frac{V_t'}{R_a + R_b \| R_c}$$

$$i_2' = \frac{v_{RB}}{R_c} = \frac{R_b \| R_c}{R_a + R_b \| R_c} \frac{V_t'}{R_c}$$

$$R_1 = \frac{V_t'}{i_1'} = R_a + R_b \| R_c = 1095\Omega \qquad \frac{R_2}{a_{12}} = \frac{V_t'}{i_2'} = R_c \frac{R_a + R_b \| R_c}{R_b \| R_c} = 1150\Omega$$

2. Apply a test voltage at v_2. Short circuit v_1. Calculate the terminal currents.



$$i_2'' = \frac{V_t''}{R_{eq}} = \frac{V_t'}{R_c + R_a \| R_b}$$

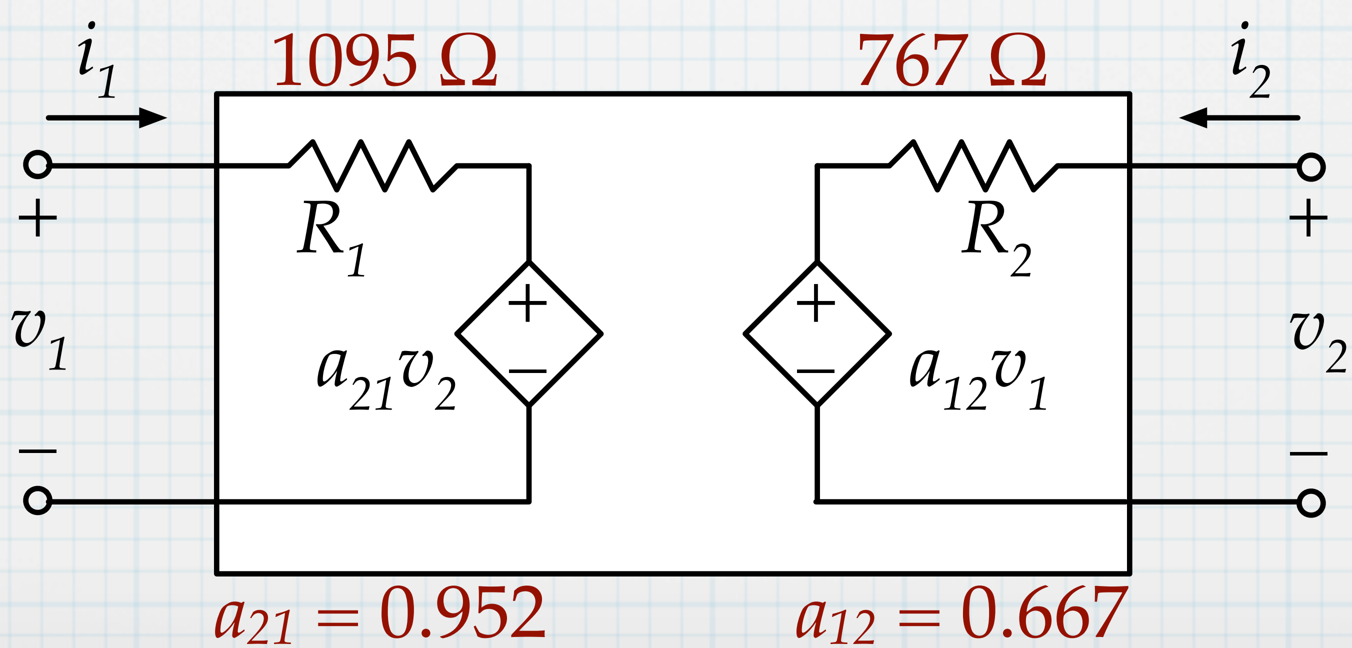
$$i_1'' = \frac{v_{Rb}}{R_a} = \frac{R_a \| R_b}{R_c + R_a \| R_b} \frac{V_t''}{R_a}$$

$$R_2 = \frac{V_t''}{i_2''} = R_c + R_a \| R_b = 767\Omega$$

$$\frac{R_1}{a_{21}} = \frac{V_t''}{i_1''} = R_a \frac{R_c + R_a \| R_b}{R_a \| R_b} = 1150\Omega$$

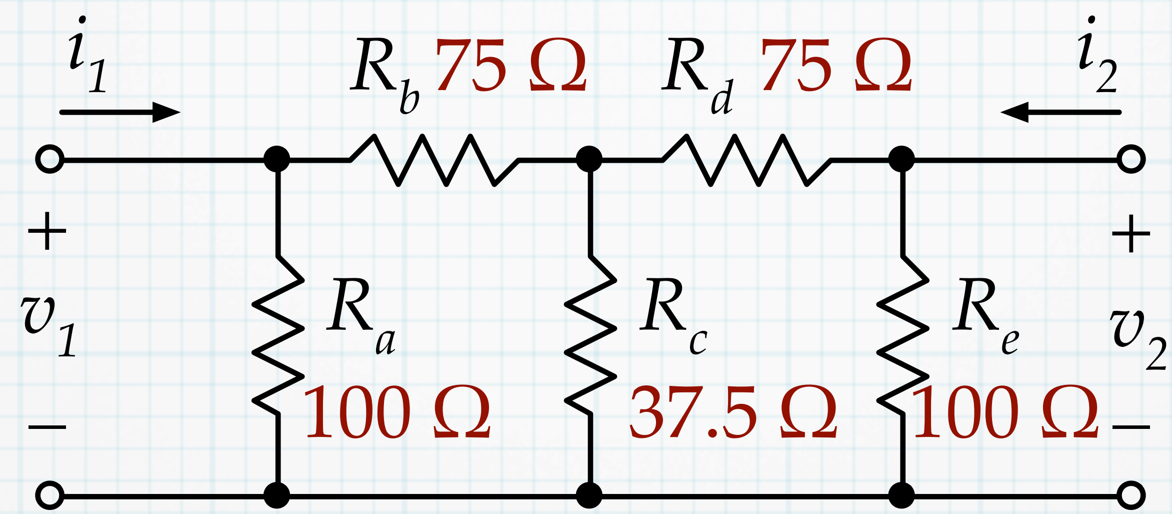
$$a_{12} = \frac{R_2}{R_2 / a_{12}} = \frac{767\Omega}{1150\Omega} = 0.667$$

$$a_{21} = \frac{R_1}{R_1 / a_{21}} = \frac{1095\Omega}{1150\Omega} = 0.952$$

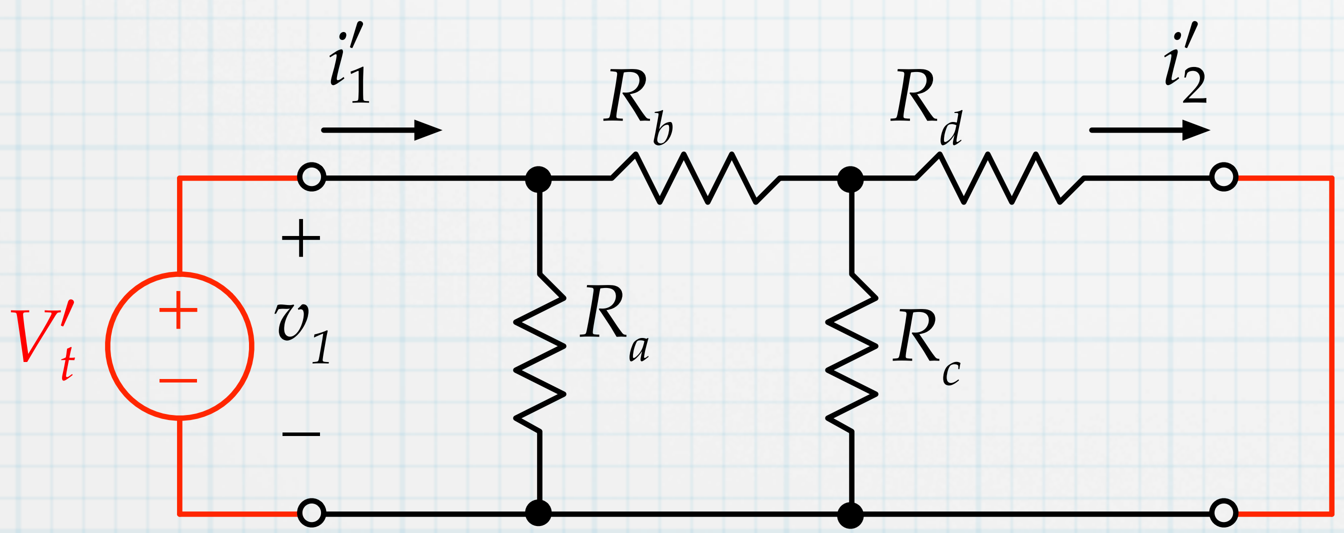


Example 2

Calculate the two-port parameters for the circuit. Note the symmetry.



1. Apply a test voltage at v_1. Short circuit v_2. (R_e is shorted out.) Calculate the terminal currents.

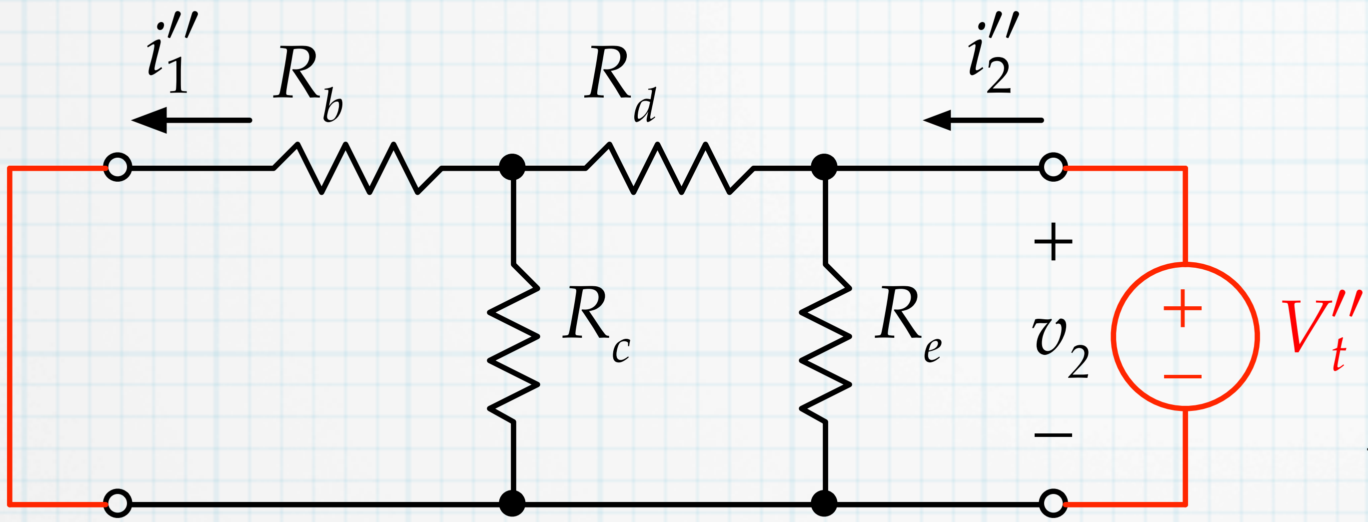


$$i'_1 = \frac{V'_t}{R_{eq}} = \frac{V'_t}{R_a \| \left[R_b + R_c \| R_d\right]}$$

$$R_1 = \frac{V'_t}{i'_2} = R_a \| \left[R_b + R_c \| R_d\right] = 50\Omega$$

$$i'_2 = \frac{v_{Rc}}{R_d} = \frac{R_c \| R_d}{R_b + R_c \| R_d} \frac{V'_t}{R_d} \qquad \frac{R_2}{a_{12}} = \frac{V'_t}{i'_2} = R_d \frac{R_b + R_c \| R_d}{R_c \| R_d} = 300\Omega$$

2. Short circuit v_1. Apply a test voltage at v_2. Calculate the terminal currents.



$$i_2'' = \frac{V_t'}{R_{eq}} = \frac{V_t''}{R_e \| \left[R_d + R_c \| R_b\right]}$$

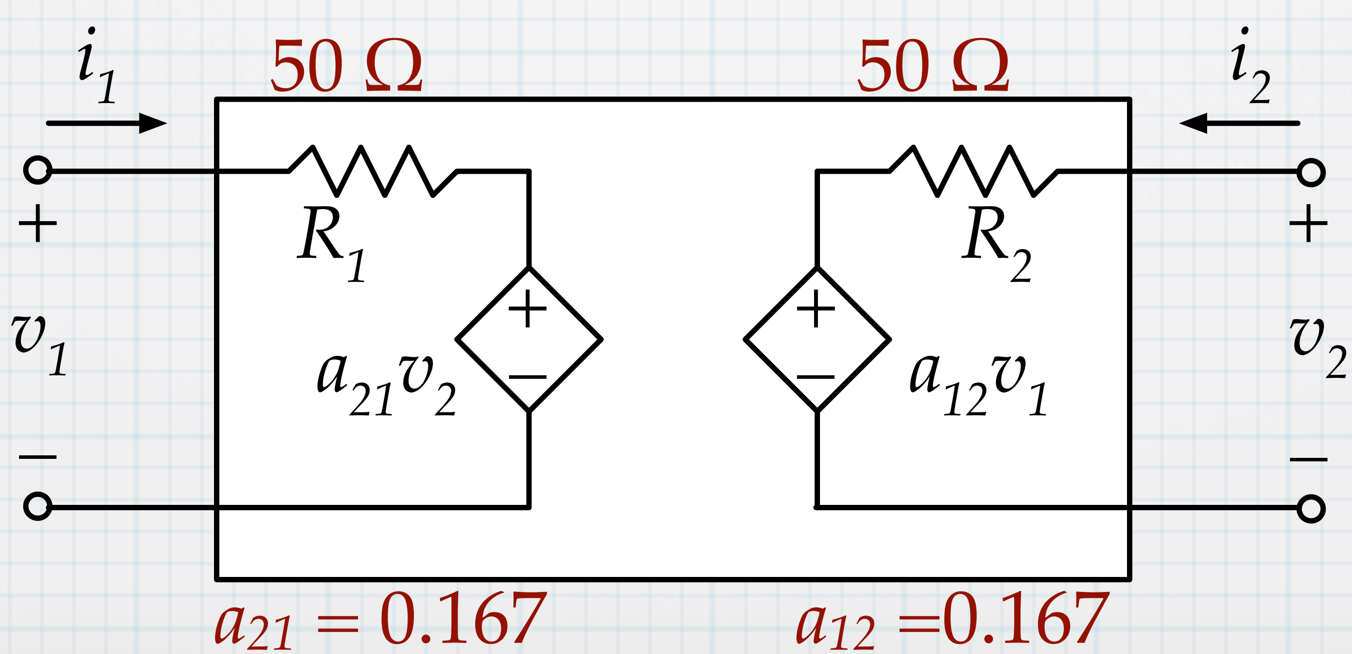
$$R_2 = \frac{V_t''}{i_2''} = R_e \| \left[R_d + R_c \| R_a\right] = 50\Omega$$

$$i_1'' = \frac{v_{Rc}}{R_b} = \frac{R_c \| R_b}{R_d + R_c \| R_b}\frac{V_t''}{R_b}$$

$$\frac{R_1}{a_{21}} = \frac{V_t''}{i_1''} = R_b\frac{R_d + R_c \| R_b}{R_c \| R_b} = 300\Omega$$

$$a_{21} = \frac{R_1}{R_1/a_{21}} = \frac{50\Omega}{300\Omega} = 0.167$$

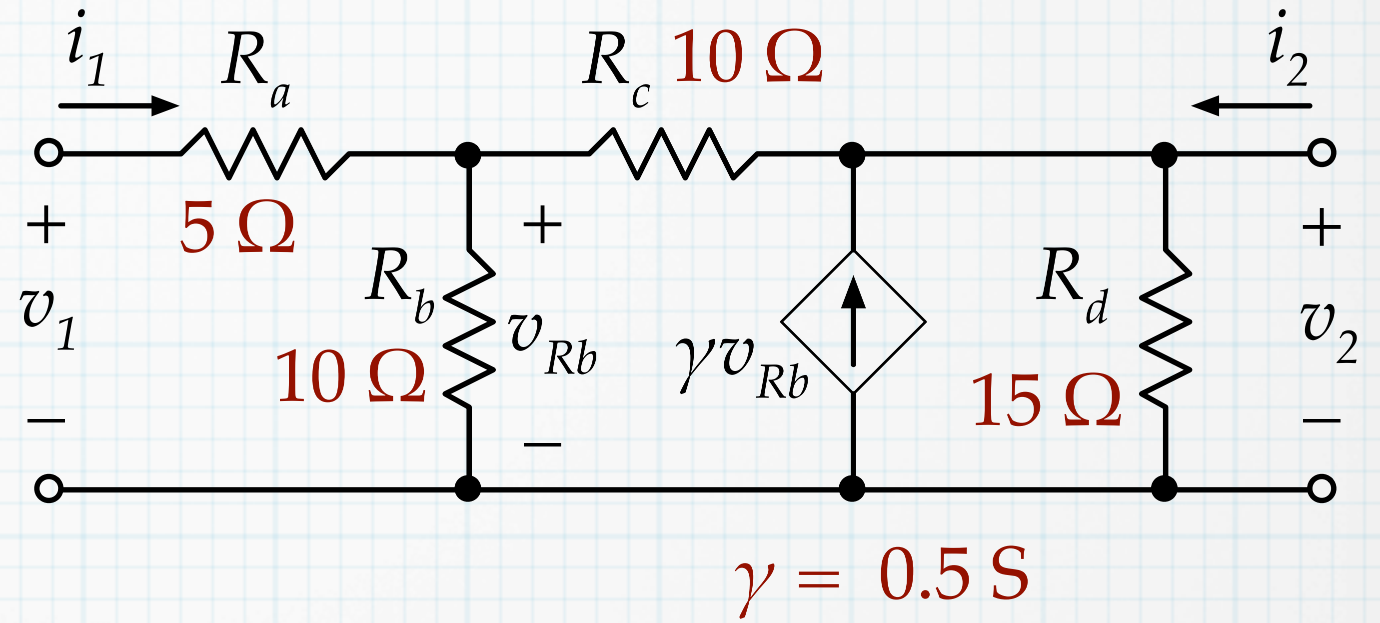
$$a_{12} = \frac{R_2}{R_2/a_{12}} = \frac{50\Omega}{300\Omega} = 0.167$$



As expected, also symmetric

Example 3

Calculate the two-port parameters for the circuit. Note that the dependent source will probably make this circuit asymmetric.



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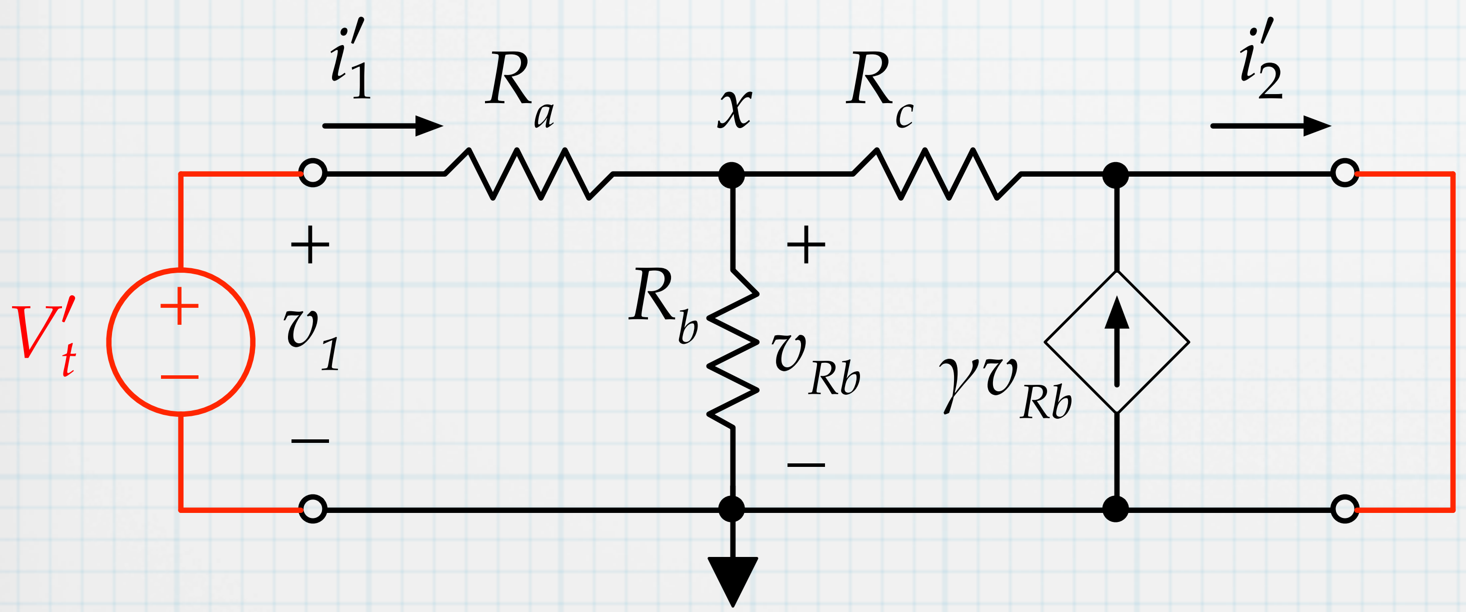
1. Short circuit v_2. (R_d is shorted out.) Apply a test voltage at v_1. Calculate the terminal currents.

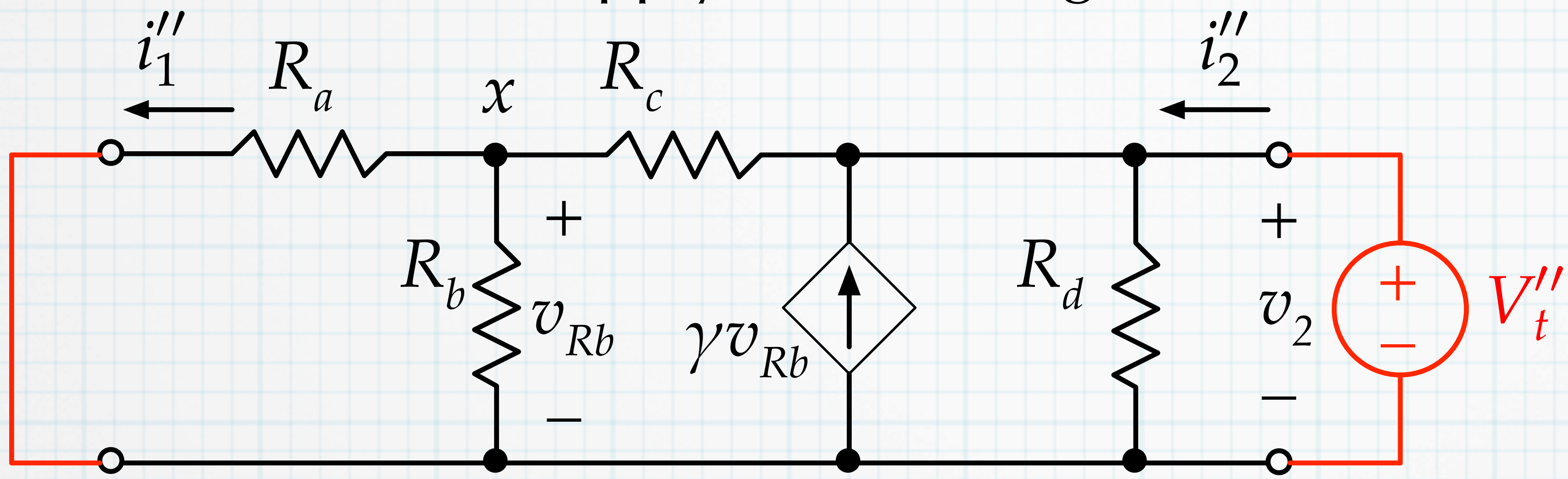


$$\frac{V_t' - v_x}{R_a} = \frac{v_x}{R_b} + \frac{v_x}{R_c} \qquad v_x = \frac{V_t'}{1 + \frac{R_a}{R_b} + \frac{R_a}{R_c}} = \frac{V_t'}{2}$$

$$i_1' = \frac{V_t' - v_x}{R_a} = \frac{V_t'}{10\Omega}$$

$$\begin{aligned} i_2' &= i_{Rc} + \gamma v_{Rb} \\ &= \frac{v_x}{R_c} + \gamma v_x \\ &= \frac{V_t'}{2R_c} + \frac{\gamma}{2} V_t' = \frac{V_t'}{3.333\Omega} \end{aligned}$$

2. Short circuit v_1. Apply a test voltage at v_2. Calculate the terminal currents.



$$i_1'' = \frac{v_x}{R_a} = \frac{V_t''}{20\Omega}$$

$$i_2'' + \gamma v_{Rb} = i_{Rd} + i_{Rc}$$

$$i_2'' + \gamma v_x = \frac{V_t''}{R_d} + \frac{V_t'' - v_x}{R_c}$$

$$\frac{V_t'' - v_x}{R_c} = \frac{v_x}{R_a} + \frac{v_x}{R_b}$$

$$i_2'' = \left[\frac{1}{R_d} + \frac{3}{4R_c} - \frac{\gamma}{4}\right] V_t'' = \frac{V_t''}{60\Omega}$$

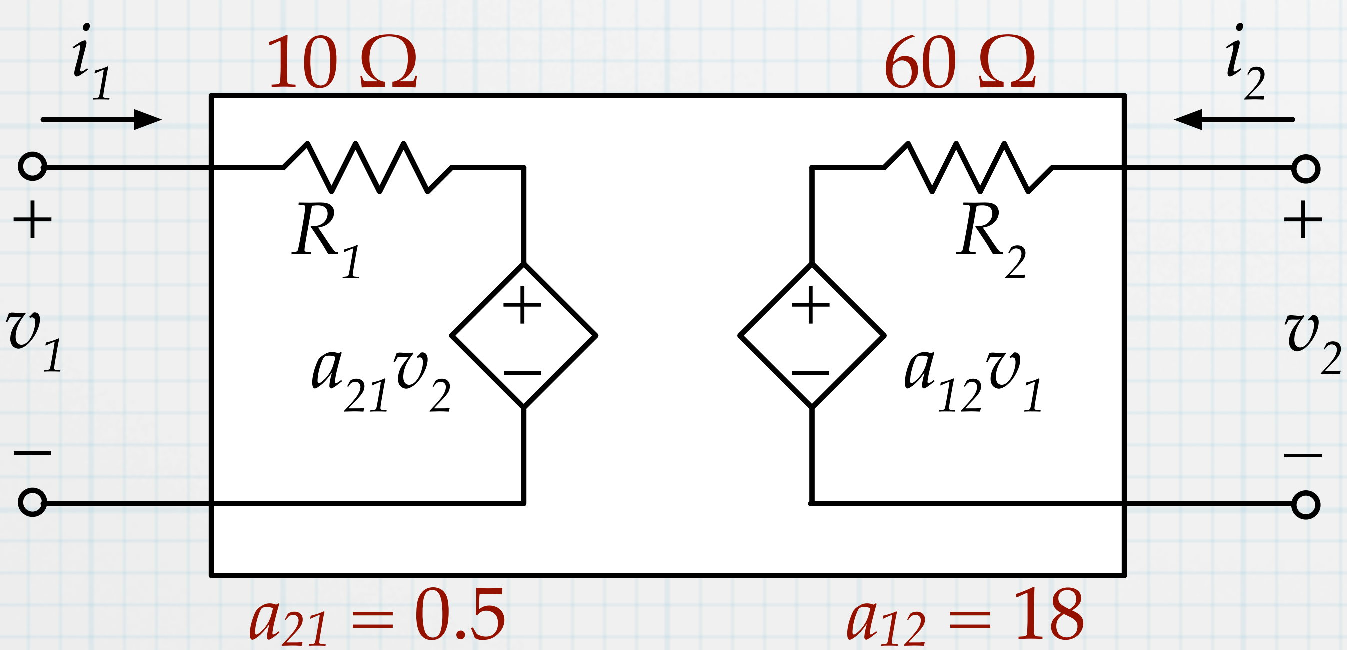
$$v_x = \frac{V_t''}{1 + \frac{R_c}{R_a} + \frac{R_c}{R_b}} = \frac{V_t''}{4}$$

$$R_1 = \frac{V_t'}{i_1'} = 10\Omega$$

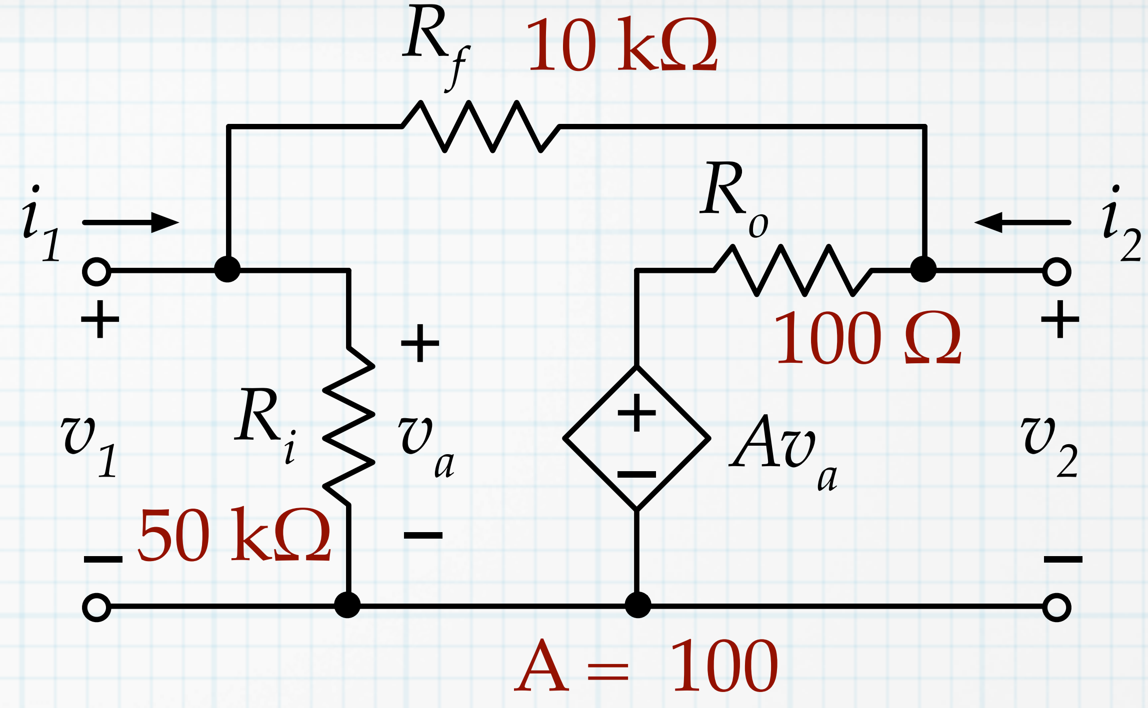
$$R_2 = \frac{V_t''}{i_2''} = 60\Omega$$

$$a_{12} = \frac{R_2}{V_t'/i_2'} = 18$$

$$a_{21} = \frac{R_1}{V_t''/i_1''} = 0.5$$

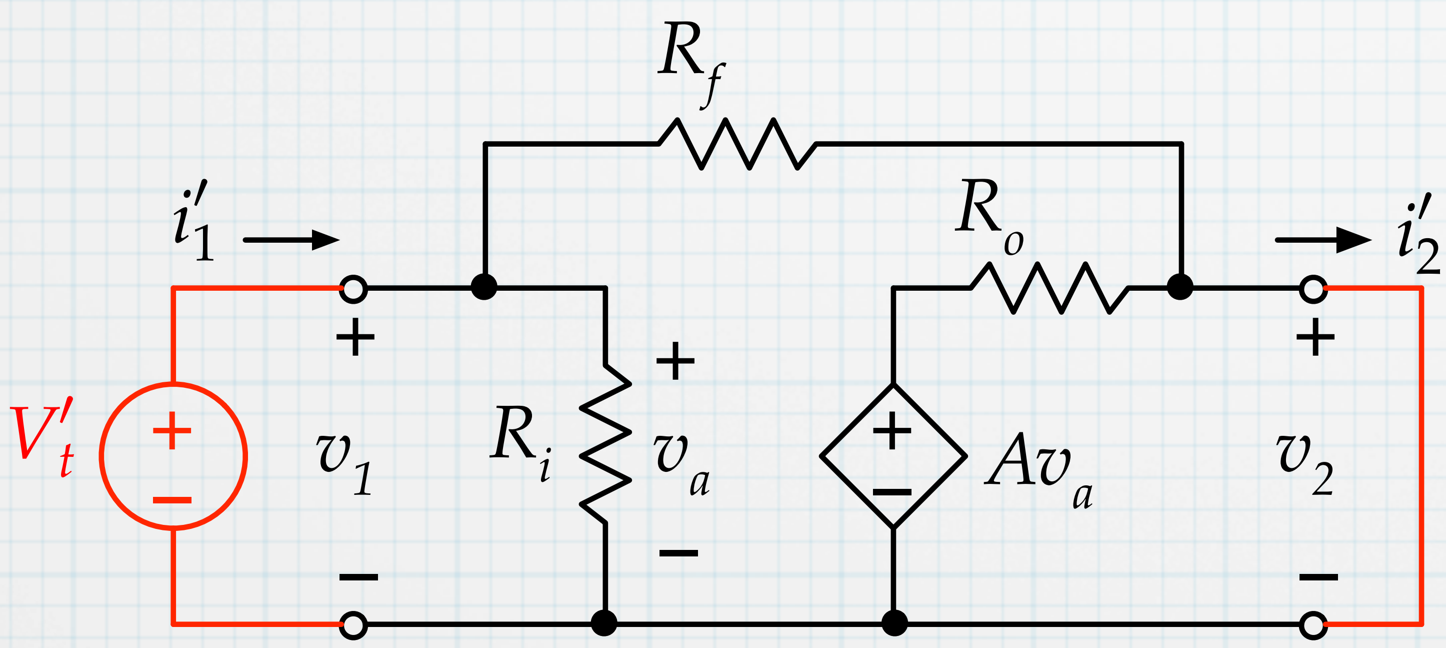


Example 4



Calculate the two-port parameters for the circuit.

1. Apply a test voltage at v_1. Short circuit v_2. Calculate the terminal currents.



$$i'_1 = \frac{V'_t}{R_i} + \frac{V'_t}{R_f}$$

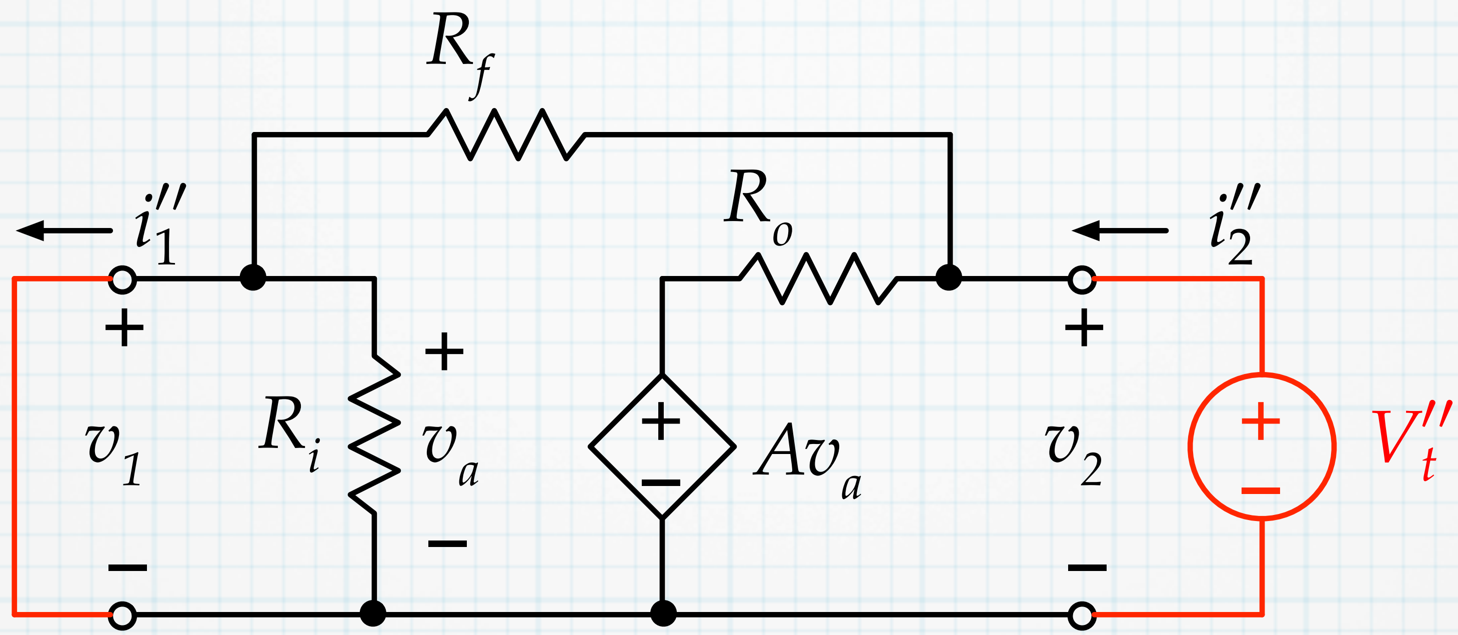
$$i'_2 = \frac{Av_a}{R_o} + \frac{V'_t}{R_f}$$

$$= \frac{AV'_t}{R_o} + \frac{V'_t}{R_f}$$

$$\frac{V'_t}{i'_1} = R_i || R_f = R_1$$

$$\frac{V'_t}{i'_2} = \left(\frac{R_o}{A}\right) \Bigg|\Bigg| R_f = \frac{R_2}{a_{12}}$$

2. Apply a test voltage at v_2. Short circuit v_1. Calculate the terminal currents.



$$i_1'' = \frac{V_t''}{R_f}$$

$$\frac{V_t''}{i_2''} = R_o||R_f = R_2$$

R_i shorted, $v_a = 0$.

$$i_2' = \frac{V_t''}{R_o} + \frac{V_t''}{R_f}$$

$$\frac{V_t''}{i_1''} = R_f = \frac{R_1}{a_{21}}$$

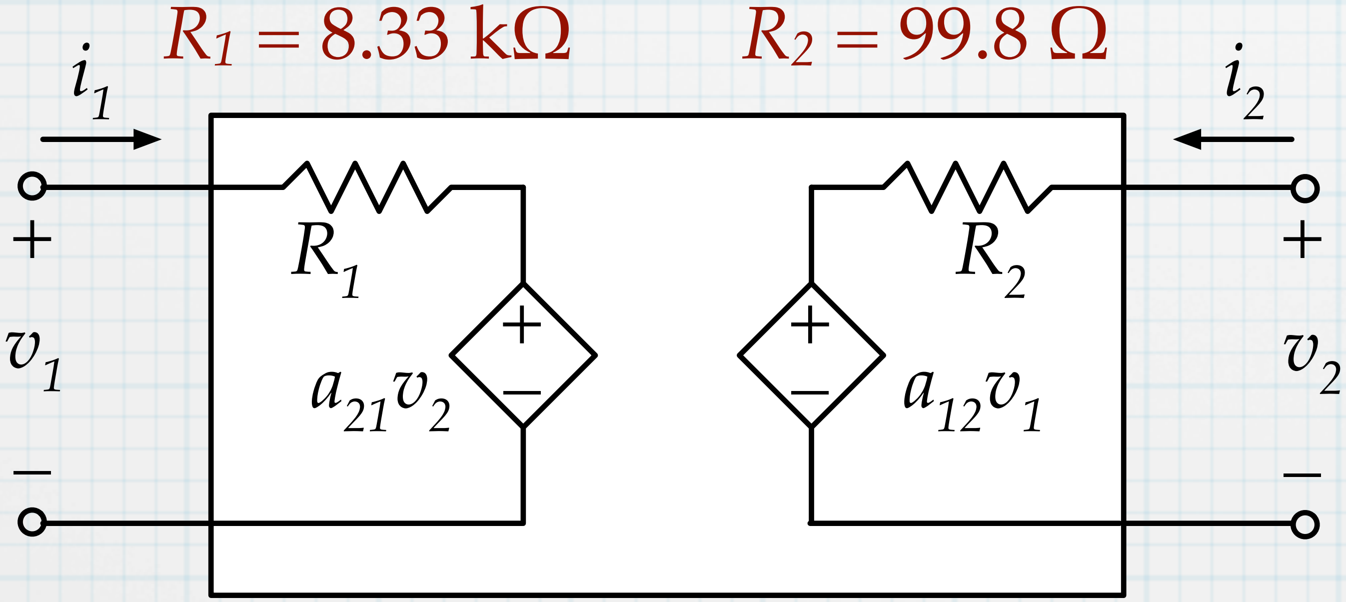
$$R_1 = \frac{V_t'}{i_1'} = R_i||R_f \quad = 8.33\ \text{k}\Omega$$

$$R_2 = \frac{V_t''}{i_2''} = R_o||R_f \quad = 99.8\ \Omega$$

$$a_{12} = \frac{R_2}{V_t'/i_2'} = \frac{R_o||R_f}{\left(\frac{R_o}{A}\right)\Big|\Big|R_f} \quad = 99.0$$

$$a_{21} = \frac{R_1}{V_t''/i_1''} = \frac{R_i||R_f}{R_f} \quad = 0.833$$

$R_1 = 8.33$ kΩ $\quad R_2 = 99.8$ Ω



$a_{21} = 0.833 \quad a_{12} = 99$

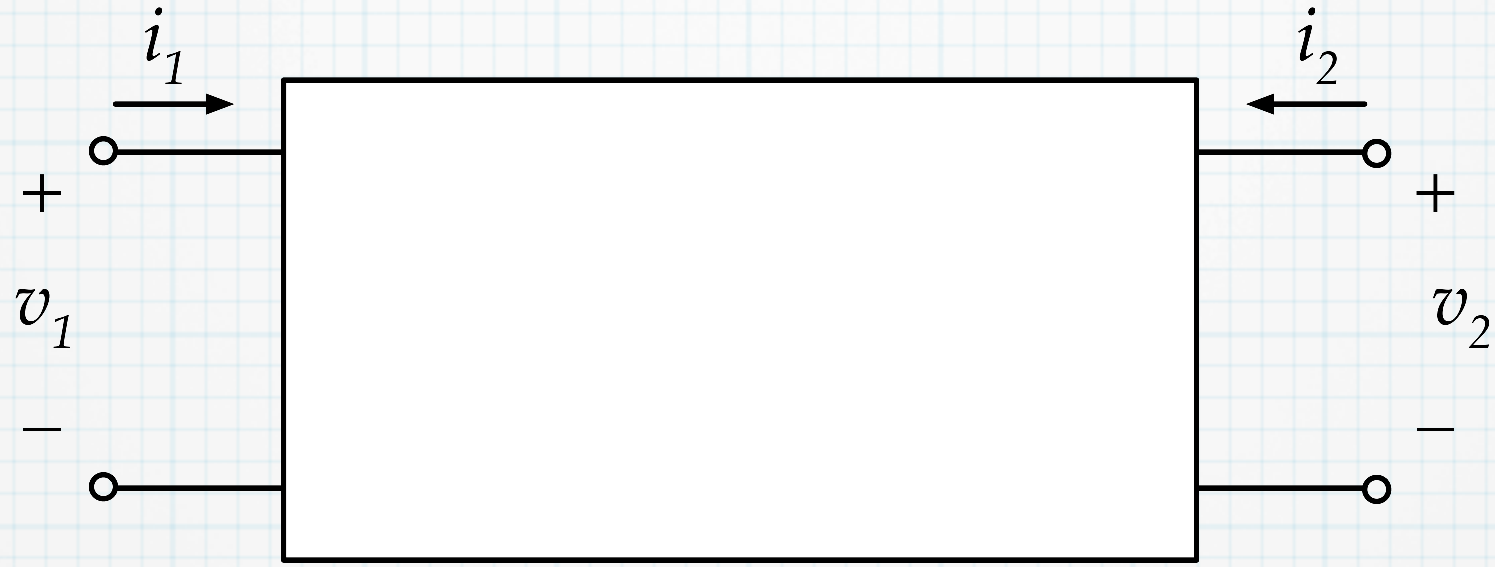
To summarize:

1. Apply a test voltage, V_t', at port 1 so that $v_1 = V_t'$.
2. Short the terminals at port 2, making $v_2 = 0$. This has the effect of making the source $a_{21}v_2 = 0$.
3. Measure, (or calculate), the currents at the two ports. The two measurements (calculations) give R_1 directly and the ratio R_2/a_{12}.
4. Apply a test voltage, V_t'' at port 2 so that $v_2 = V_t''$.
5. Short the terminals at port 1, making $v_1 = 0$. This has the effect of making the source $a_{12}v_1 = 0$.
6. Measure, or calculate, the currents at the two ports. The two measurements (calculations) give R_2 directly and the ratio R_1/a_{21}.
7. Calculate a_{12} and a_{21} from the acquired data.

Note: If a circuit is symmetric, it's two-port equivalent will also be symmetric.

Note: If there are no dependent sources, only resistors, the circuit is *passive* and $a_{12} < 1$ and $a_{21} < 1$. If there are dependent sources, then circuit may be *active*, such that either $a_{12} > 1$ or $a_{21} > 1$ or both.

General two-port theory



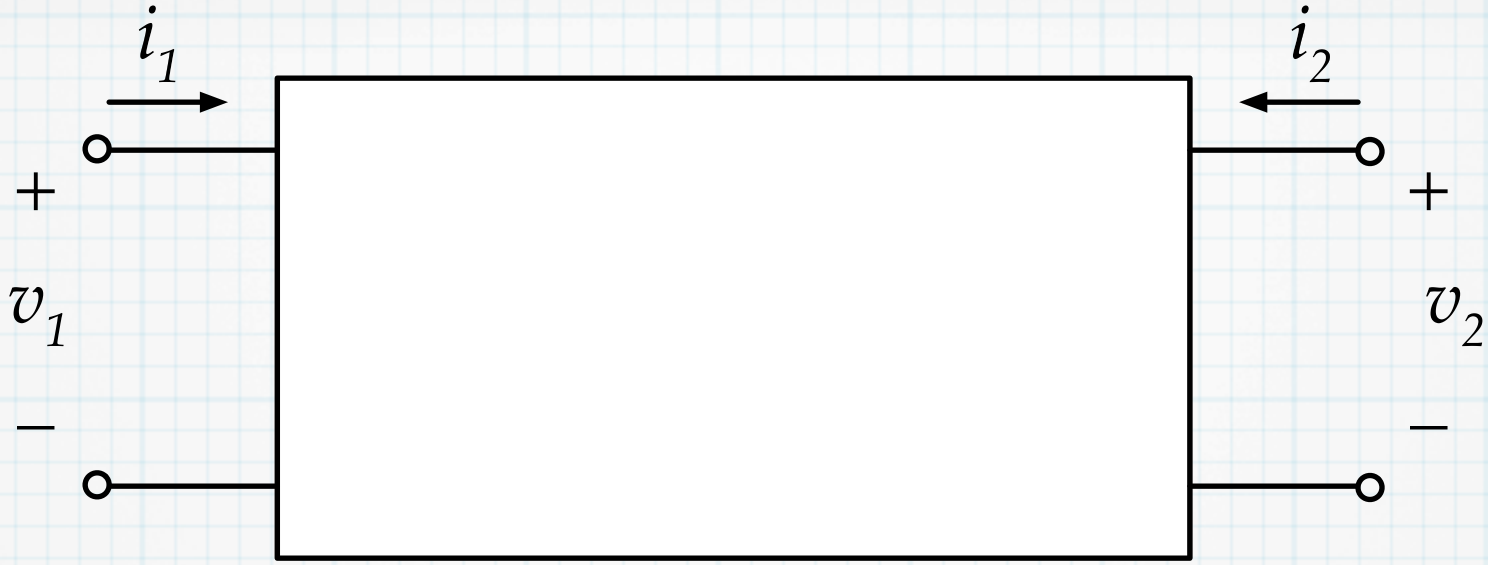
The form of two-port equivalent that we have used to this point is just one of several different forms. There are other ways to view the relationship between the terminal currents and voltages. What we have done is to write the currents in terms of the voltages.

$$i_1 = y_{11}v_1 + y_{12}v_2$$
$$i_2 = y_{21}v_1 + y_{22}v_2$$

$$y_{11} = \left.\frac{i_1}{v_1}\right|_{v_2=0} \qquad y_{12} = \left.\frac{i_1}{v_2}\right|_{v_1=0}$$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$y_{21} = \left.\frac{i_2}{v_1}\right|_{v_2=0} \qquad y_{22} = \left.\frac{i_2}{v_2}\right|_{v_1=0}$$



Express v in terms of i (impedance parameters)

$$v_1 = z_{11}i_1 + z_{12}i_2$$
$$v_2 = z_{21}i_1 + z_{22}i_2$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Units of Ω for all parameters.

Express output in terms of input (transmission parameters)

$$v_2 = t_{11}v_1 + t_{12}i_1$$
$$i_2 = t_{21}v_1 + t_{22}i_1$$

$$\begin{bmatrix} v_2 \\ i_2 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ i_1 \end{bmatrix}$$

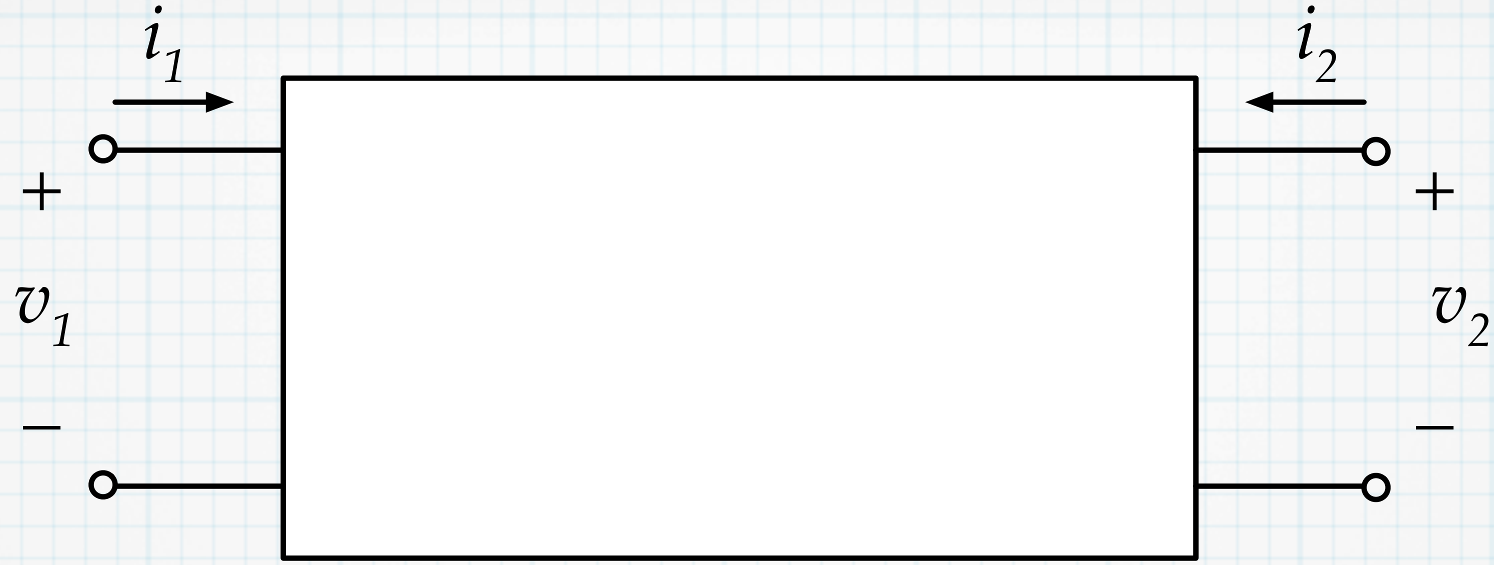
Different units for different parameters.

Express input in terms of output (reflection parameters)

$$v_1 = r_{11}v_2 + r_{12}i_2$$
$$i_1 = r_{21}v_2 + r_{22}i_2$$

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix}$$

Different units for different parameters.



Express v_1 and i_2 terms of v_2 and i_1 (hybrid parameters)

$$v_1 = h_{11} v_2 + h_{12} i_1$$
$$i_2 = h_{21} v_2 + h_{22} i_1$$

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} v_2 \\ i_1 \end{bmatrix}$$

Different units for different parameters.

Express v_2 and i_1 terms of v_1 and i_2 (also known as hybrid parameters)

$$v_2 = g_{11} v_1 + g_{12} i_2$$
$$i_1 = g_{21} v_1 + g_{22} i_2$$

$$\begin{bmatrix} v_2 \\ i_1 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

Different units for different parameters.

With these different forms, the *i*-*v* relationships become more abstract. We do not even need to know about the specific components making up the equivalent two-port network. The matrix elements tell us everything we need.

If we know set of matrix parameters, we can convert to any other set. See the giant conversion tables in the text book. (For math nerds in the class: derive the conversion formulas. For example, convert *y*-parameters to *t*-parameters. It's not hard, just tedious linear algebra.)

For the most part, we will only use *y*-parameters to express the typical equivalent circuit for amplifiers. With that starting point, we can shift to other forms using source transformations, if needed.