Complex numbers

The need for imaginary and complex numbers arises when finding the two roots of a quadratic equation.

$$\alpha x^2 + \beta x + \gamma = 0$$

The two roots are given by the quadratic formula

$$x = -\frac{\beta}{2\alpha} \pm \sqrt{\left(\frac{\beta}{2\alpha}\right)^2 - \frac{\gamma}{\alpha}}$$

There are no problems as long as $(\beta/2\alpha)^2 \geq \gamma/\alpha$ – there are two real roots and everything is clean. But if $(\beta/2\alpha)^2 < \gamma/\alpha$, then we are faced with having to take the square-root of a negative number.

In "ancient" times, such situations were deemed impossible and simply ignored. And yet, physical systems described by the "impossible" parameters continued to function, generally with very interesting results. Clearly, ignoring the problem is not helpful.

So what to do when faced with such situations?

$$z = a + \sqrt{-b^2}$$

It took a couple of hundred years, but the people working on the problem realized that the square-root term had useful physical information and could not be ignored. However, square-root term was different from the real number represented by the first term. The second term had to be treated in a special way, and a new algebra had to be developed to handle these special numbers. (Actually, the *new* algebra is an extension of the *old* real number algebra.

The special nature of the square-root term is signified by introducing a new symbol.

$$\sqrt{-b^2} = \sqrt{-1}\sqrt{b^2} = jb$$

where $j = \sqrt{-1}$ and b is conventional real number.

(Note: In almost all other fields, it is conventional to use $i = \sqrt{-1}$. However, in EE/CprE, we use i for current, and so it has become normal practice in our business to use j.)

Clearly, this number j has some interesting properties:

$$j \cdot j = j^2 = -1.$$

$$j^3 = j \cdot j \cdot j = (j \cdot j) \cdot j = (-1) \cdot j = -j.$$

$$j^4 = j^2 \cdot j^2 = (-1) \cdot (-1) = +1.$$

$$j^5 = j^4 \cdot j = (+1) \cdot j = +j.$$

Looking at successively higher powers of j, we cycle through the four values, $+j, -1, -j, +1$.

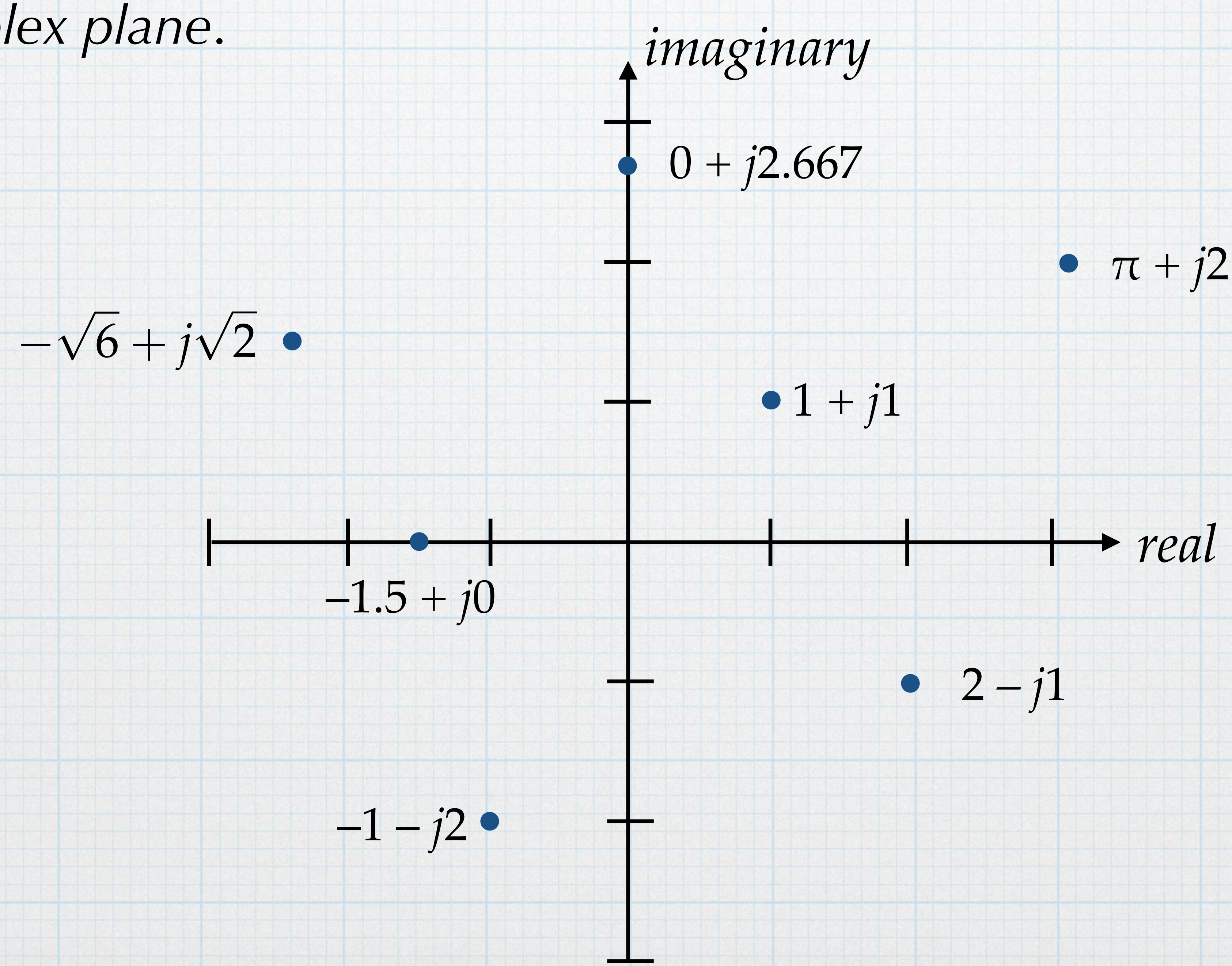
A number, like jb, that has a negative value for its square, is known as an *imaginary* number. (This is really a poor choice of terminology.)

A number, like $z = a + jb$, that is the sum of a real term and an imaginary term is known as a *complex* number.

How to work with this new type of number? Clearly, an imaginary number is somehow different from a familiar real number. In thinking about how real numbers relate to each other and when visualizing functions of real numbers, we often start with a real number line. All real numbers are represented by a point on the line. Similarly, imaginary numbers can be represented by points on an imaginary number line.

Complex number plane

Now we have two number lines – how are they related? In working this out, the early mathematicians came to the conclusion that the imaginary axis is perpendicular to the real axis, so that the two axes form what is essentially an x-y set of coordinates. The real and imaginary parts of a complex number give the coordinates of a point in the *complex plane*.



Complex math – addition and subtraction

Addition and subtraction with complex numbers is straight-forward. Add (or subtract) the real parts and then add (or subtract) the imaginary parts. Obviously, the result is also a complex number.

$$z_1 = a + jb \qquad z_2 = c + jd$$

$$z_1 + z_2 = (a + jb) + (c + jd) = (a + c) + j(b + d)$$

$$z_1 - z_2 = (a + jb) - (c + jd) = (a - c) + j(b - d)$$

$$(1 + j4) + (2 + j1) = 3 + j5 \qquad (1 + j4) - (2 + j1) = -1 + j3$$

$$(-1 + j4) + (2 - j6) = 1 - j2 \qquad (-1 + j4) - (2 - j6) = -3 + j10$$

Complex math – multiplication

Multiplication is also straight-forward. It is essentially the same as multiplying polynomials — just make sure that every term is multiplied by every other term. The result will be a mixing of the reals and imaginaries from the two factors, and these will need to be sorted out for the final result.

$$z_1 \cdot z_2 = (a + jb)\cdot(c + jd) = ac + jad + jbc + (j)^2 bd$$

Note that the two imaginary terms multiply together to give a real, since $j^2 = -1$. Collect the real and imaginary parts to write the complex number in standard form.

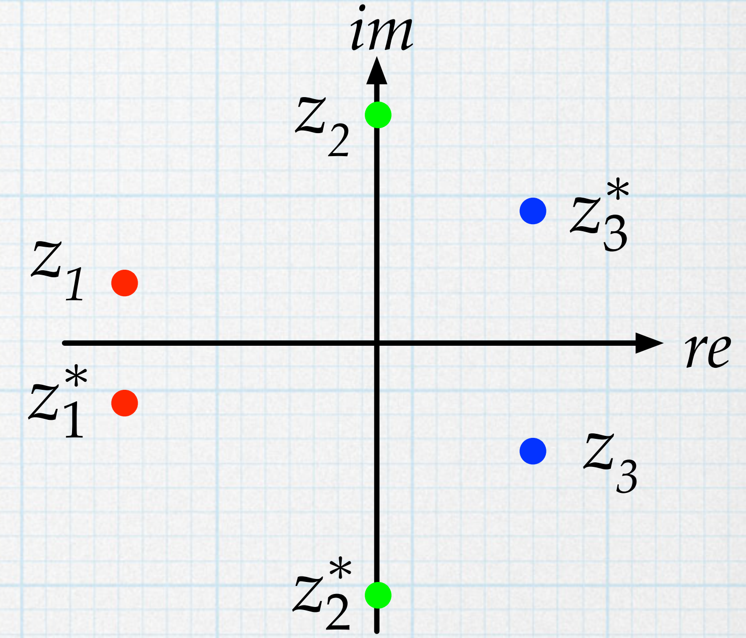
$$z_1 \cdot z_2 = (a + jb)\cdot(c + jd) = (ac - bd) + j(ad + bc)$$

Complex math – complex conjugates

The two roots that are the solutions to a quadratic equation may be complex. In that case, the roots come as set:

$$z_1 = a + jb \quad \text{and} \quad z_2 = a - jb$$

The same real part and the imaginary parts have opposite signs.



Conjugates in the complex plane.

Numbers having this relationship are known as *complex conjugates*. Every complex number, z, has a conjugate, denoted as z^*. From above

$$z_1^* = a - jb \quad \text{and} \quad z_2^* = a + jb$$

Again, the two roots are complex conjugates of each other.

$$z \cdot z^* = (a + jb) \cdot (a - jb) = a^2 - jab + jab + b^2 = a^2 + b^2$$ purely real!

$$z + z^* = (a + jb) + (a - jb) = 2a$$ purely real

$$z - z^* = (a + jb) - (a - jb) = j\,(2b)$$ purely imaginary

Complex math – division

Dividing one complex number by another gets messier.

$$\frac{z_1}{z_2} = \frac{a + jb}{c + jd}$$

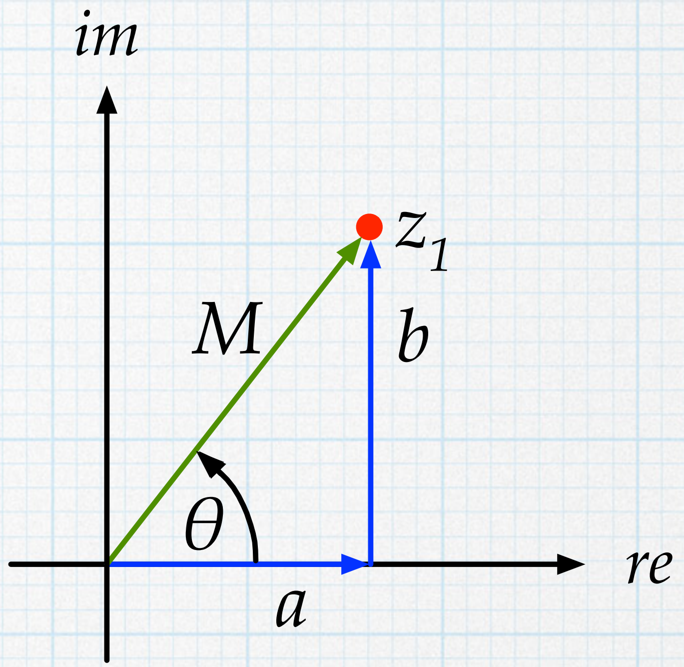
It looks like we would have to resort to methods used when dividing polynomials. But we are saved with a trick using complex conjugates. Recall that when a complex number is multiplied by it's conjugate, the result is a purely real number. Making use of that, we multiply numerator and denominator by $z_2{}^*$.

$$\frac{z_1}{z_2} \cdot \frac{z_2^*}{z_2^*} = \frac{a + jb}{c + jd} \cdot \frac{c - jd}{c - jd}$$

$$\frac{z_1}{z_2} = \frac{ac - jad + jbc + bd}{c^2 + d^2} = \underbrace{\left(\frac{ac + bd}{c^2 + d^2} \right)}_{re} + j \underbrace{\left(\frac{bc - ad}{c^2 + d^2} \right)}_{im}$$

Polar representation

Specifying a complex number in the complex plane using the real and imaginary parts is quite simple — it the same as specifying points with rectangular coordinates.



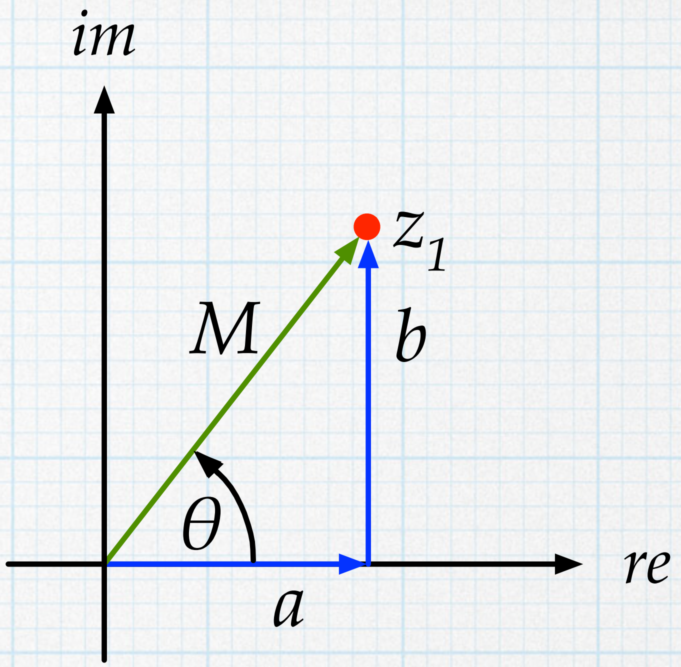
However, we also know that a point can be specified using polar coordinates. In our case, we would locate the complex number in the plane by specifying an angle (or heading) and distance from the origin to the point along that heading. To describe the complex number in polar form, we use the magnitude (M) and the angle (θ). A commonly used notation for specifying a complex number in polar form is to list the magnitude followed by the angle inside a "angle bracket" simple.

$$z_1 = M\underline{/\theta}$$

You might see this notation in many circuits texts. We will not use it in EE 201, because there is a better notation that is more descriptive.

rectangular to polar (and back)

From the plot in the complex plane, we see that the conversion from rectangular form ($a + jb$) to polar form ($M\angle\theta$) is a simple application of trigonometry.



$$M = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

It is equally easy to convert from polar to rectangular.

$$a = M\cos\theta$$

$$b = M\sin\theta$$

Euler

One of the more profound notions in math is that if that if we take the exponential of an imaginary angle, exp($j\theta$) the result is a complex number. The interpretation is given by *Euler's formula*.

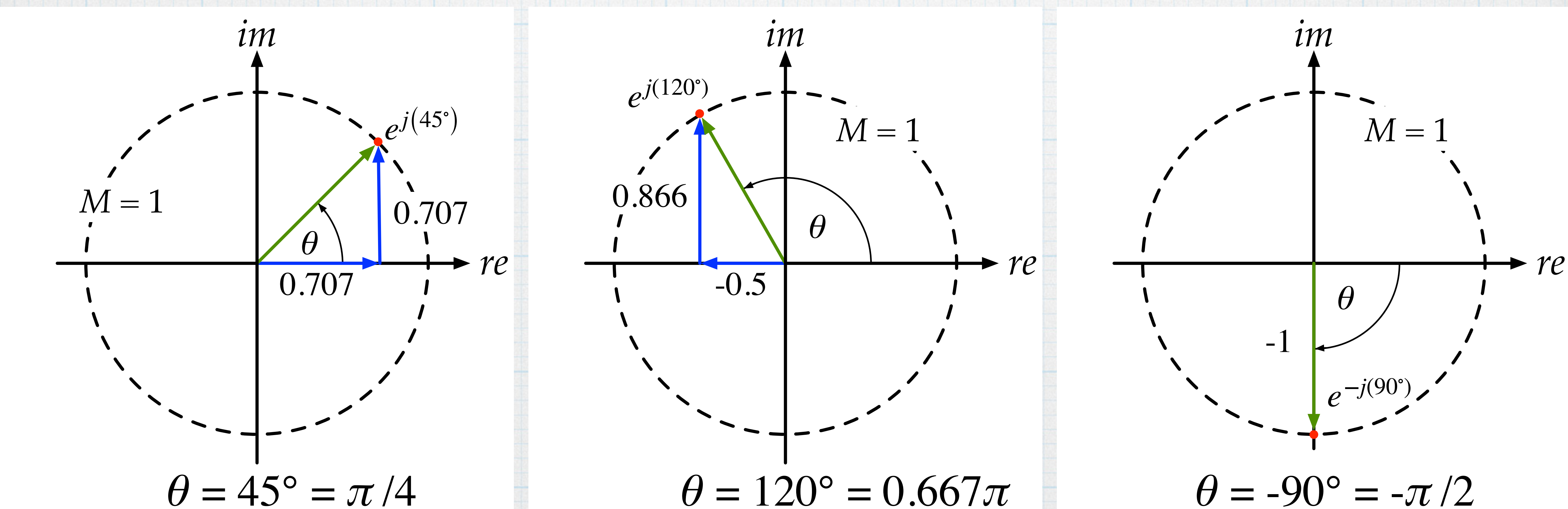
$$\exp\left(j\theta\right) = \cos\theta + j\sin\theta = a + jb$$

$$\exp\left(j\pi\right) + 1 = 0$$

(Euler's identity.)

Every complex number of this form has a magnitude of 1.

$$M = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$



pseudo-proof

$$\exp(\theta) = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \frac{\theta^6}{6!} + \frac{\theta^7}{7!} + \ldots$$

$$= 1 + j\frac{\theta}{1!} - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - j\frac{\theta^7}{7!} + \ldots$$

$$= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right] + j\left[\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right]$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \qquad \sin\theta = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$$

$$\cos\theta + j\sin\theta = \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right] + j\left[\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right]$$

$$\cos\theta + j\sin\theta = \exp(j\theta)$$

The expression exp($j\theta$) is a complex number pointing at an angle of θ and with a magnitude of 1. (M = 1). We can use this notation to express other complex numbers with M ≠ 1 by multiplying by the magnitude.

$$z = M \exp\left(j\theta\right)$$

Using Euler's formula:

$$z = M \exp\left(j\theta\right) = M\cos\theta + jM\sin\theta = a + jb$$

This is just another way of expressing a complex number in polar form.

$$M \exp\left(j\theta\right) \xleftrightarrow{\text{same as}} M\underline{/\theta}$$

The exponential form is actually a better representation, because it allows us to do multiplications and division directly — there's no need to convert to real/imaginary form first.

$$z_1 = M_1 \exp\left(j\theta_1\right) \qquad z_2 = M_2 \exp\left(j\theta_2\right)$$

$$z_1 \cdot z_2 = \left[M_1 \exp\left(j\theta_1\right)\right]\left[M_2 \exp\left(j\theta_2\right)\right] = \left(M_1 M_2\right) \exp\left[j\left(\theta_1 + \theta_2\right)\right]$$

Magnitudes multiply and angles add.

$$\frac{z_1}{z_2} = \frac{M_1 \exp\left(j\theta_1\right)}{M_2 \exp\left(j\theta_2\right)} = \frac{M_1}{M_2} \exp\left[j\left(\theta_1 - \theta_2\right)\right]$$

Magnitudes divide and angles subtract.

$$z_1 = 4 + j3 \longrightarrow z_1 = 5\exp\left(j36.9^\circ\right)$$

$$z_2 = 2 + j2 \longrightarrow z_2 = 2.83\exp\left(j45^\circ\right)$$

$$z_p = z_1 \cdot z_2 = \left(4 + j3\right)\left(2 + j2\right) = (4 \cdot 2 - 3 \cdot 2) + j\,(4 \cdot 2 + 3 \cdot 2) = 2 + j14$$

$$z_p = 2 + j14 \longrightarrow z_p = 14.1\exp\left(j81.9^\circ\right)$$

$$z_p = z_1 \cdot z_2 = \left[5\exp\left(j36.9^\circ\right)\right]\left[2.83\exp\left(j45^\circ\right)\right] = 14.1\exp\left(j81.9^\circ\right)$$

$$z_q = \frac{z_1}{z_2} = \frac{4 + j3}{2 + j2} = \frac{(4 \cdot 2 + 3 \cdot 2) + j\,(-4 \cdot 2 + 3 \cdot 2)}{2^2 + 2^2} = 1.75 - j0.25$$

$$z_q = 1.75 - j0.25 \longrightarrow z_q = 1.77\exp\left(-j8.1^\circ\right)$$

$$z_q = \frac{z_1}{z_2} = \frac{5\exp\left(j36.9^\circ\right)}{2.83\exp\left(j45^\circ\right)} = 1.77\exp\left(-j81.9^\circ\right)$$

The complex conjugate in polar form is also quite easy.

$$z = a + jb \longrightarrow z = M \exp\left(j\theta\right) \qquad M = \sqrt{a^2 + b^2} \qquad \theta = \arctan\left(\frac{b}{a}\right)$$

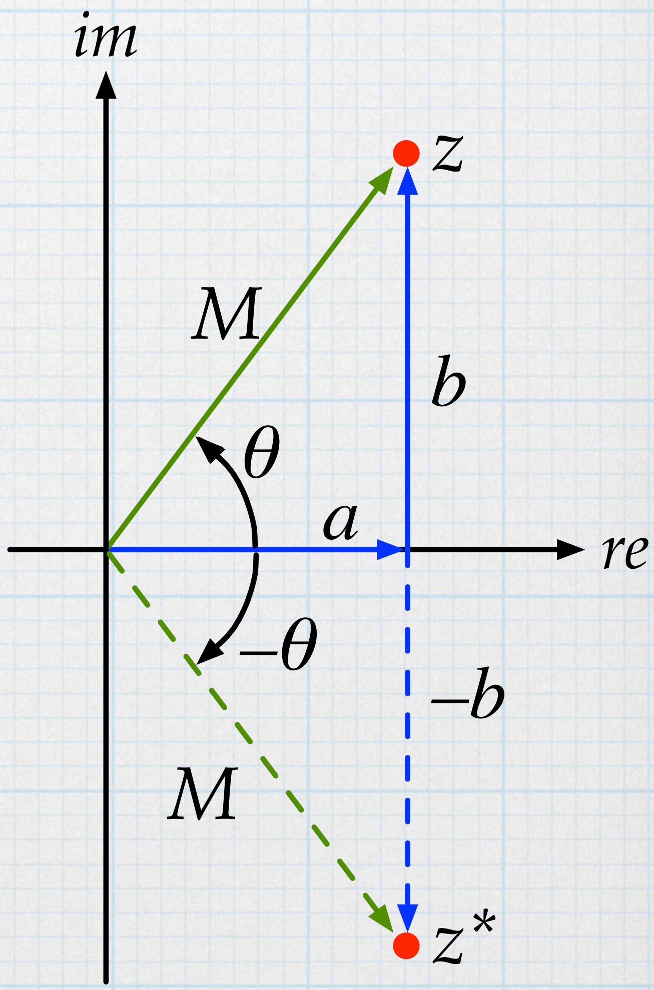
$$z* = a - jb \qquad M* = \sqrt{a^2 + b^2} = M$$

$$\theta* = \arctan\left(\frac{-b}{a}\right) = -\arctan\left(\frac{b}{a}\right) = -\theta$$

$$z* = M \exp\left(-j\theta\right)$$

$$z \cdot z* = \left[M \exp\left(j\theta\right)\right]\left[M \exp\left(-j\theta\right)\right] = M^2 = a^2 + b^2$$

As expected.



sinusoids represented as complex exponentials.

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Here is an interesting relationship: $j^{-1} = -j$

Simple proof: $j = 1 \cdot \exp\left(j\frac{\pi}{2}\right)$

$$\frac{1}{j} = \frac{1}{1 \cdot \exp\left(j\frac{\pi}{2}\right)} = 1 \cdot \exp\left(-j\frac{\pi}{2}\right) = -j$$

For a graphical interpretation, draw these in the complex plane. We will use this frequently.

Give this one a try: $\sqrt{j} = ?$