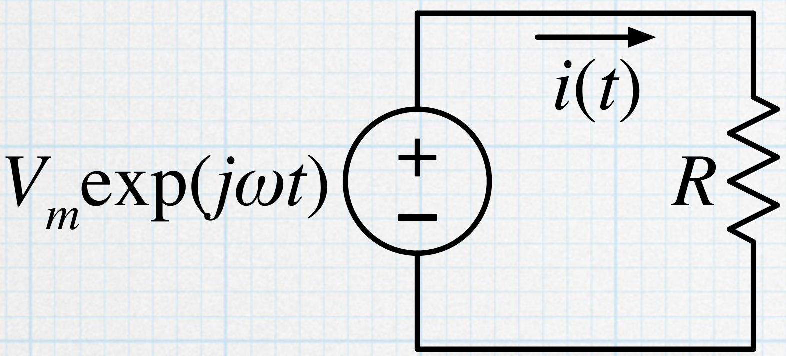
AC circuit analysis

The story so far:

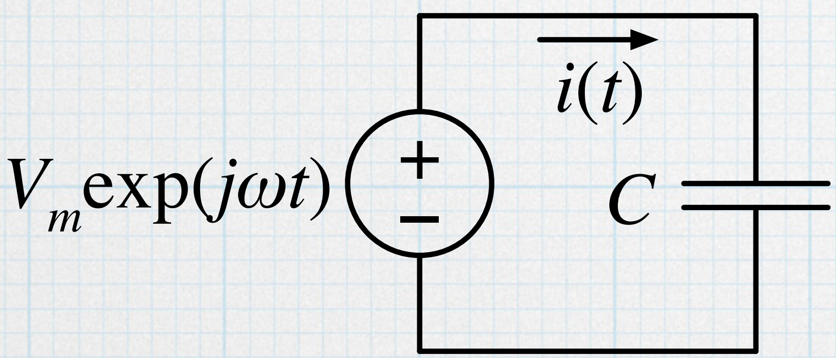
1. For circuits that are driven by sinusoidal sources (e.g. $v_s(t) = V_m \cdot \cos(\omega t)$), the voltages and currents are always sinusoids oscillating at the same frequency as the source and having distinct amplitudes. If there are capacitors and inductors in the circuit then the sinusoidal voltages and currents may also have phase shifts with respect to the source. These sinusoids, with the various amplitude and angles, are easy to see and measure in the lab.
2. Calculating the amplitudes and phase angles using conventional differential-equation techniques is messy and may involve a lot of trigonometric gymnastics.
3. If we ignore transients effects and focus on the steady-state part of the solution, the math is made easier by leaving out half of the problem. This is known as *sinusoidal steady-state analysis*.
4. If we describe the sinusoids using complex exponentials ($V_m \cdot e^{j\omega t}$) instead of sines and cosines, the math needed to solve the differential equations is easier. But there is a tradeoff — we have introduced complex numbers. Complex math is messy, but the complex form provide a compact way to describe the amplitude and phase shift information of the sinusoidal voltages and currents. They are convenient, if we learn how to use them.

But there is more to this business of using complex numbers. To see it, apply a complex exponential voltages to individual resistors, capacitors, and inductors and find expressions for the resulting currents.



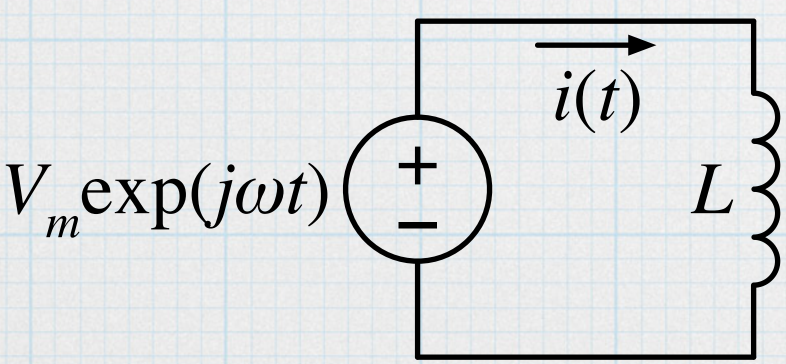
$$i_R(t) = \frac{V_m e^{j\omega t}}{R} = \frac{v_R(t)}{R}$$

$$\frac{v_R(t)}{i_R(t)} = R$$ Big deal. It's just Ohm's law.



$$i_C(t) = C\frac{d\left(V_m e^{j\omega t}\right)}{dt} = j\omega C\left[v_C(t)\right]$$

$$\frac{v_C(t)}{i_C(t)} = \frac{1}{j\omega C}$$ 👀 !!



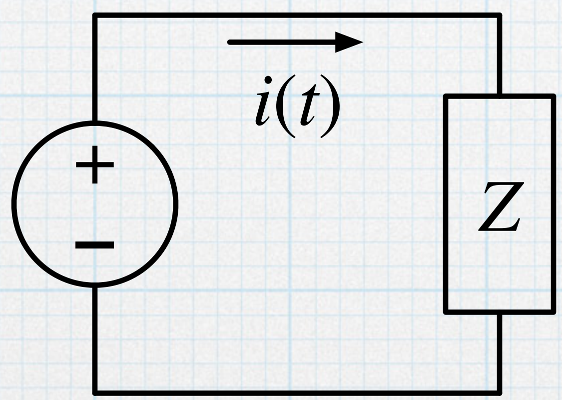
$$i_L(t) = \frac{1}{L}\int V_m e^{j\omega t} dt = \frac{v_L(t)}{j\omega L}$$

$$\frac{v_L(t)}{i_L(t)} = j\omega L$$ Wut !?!

In each case, the sinusoidal voltage results in a sinusoidal current — no big surprise here. What is surprising is that, for each component, the ratio of the of sinusoidal voltage to the sinusoidal current is a *number*.

Of course, we expect this for resistors because they obey Ohm's law, but capacitors and inductors do not follow Ohm's law. Yet, with sinusoids, there is a quantity that behaves almost like resistance. We call the quantity the *impedance*. Using impedance allows us to treat resistors, capacitors, and inductor in sinusoidal circuits in a unified manner.

$$v_S(t) = V_m \cdot e^{j\omega t}$$



$$i(t) = \frac{v_S(t)}{Z}$$

resistor: $Z = R$ capacitor: $Z = \dfrac{1}{j\omega C}$ inductor: $Z = j\omega L$

Impedance

Impedance is complex, so it carries magnitude and phase angle information.

$$\text{resistor: } i_R = \frac{v_R}{Z} = \frac{v_R}{R} = \left(\frac{v_R}{R}\right) e^{j0^\circ}$$

$$\text{capacitor: } i_C = \frac{v_C}{Z_C} = \frac{v_C}{\frac{1}{j\omega C}} = \left(j\omega C\right) v_R = \left(\omega C \cdot v_C\right) e^{j90^\circ}$$

$$\text{inductor: } i_L = \frac{v_L}{Z_L} = \frac{v_L}{j\omega L} = \left(\frac{v_L}{\omega L}\right) e^{-j90^\circ}$$

For the resistor, the current is exactly in phase with voltage. For the capacitor, the current *leads* the voltage by 90°. For the inductor, the current *lags* the voltage by 90°.

These observations are in line with what we saw when using sines and cosines to describe oscillations. The derivatives in the capacitor and inductor i-v relations turned sines to cosines and vice-versa.

Since impedance is defined as voltage divided by current, the units must be ohms.

Impedance

The capacitor and inductor impedances are purely imaginary, and are referred to generally as *reactances*. We note that inductor and capacitor impedances have opposite signs.

$$Z_L = j\omega L$$

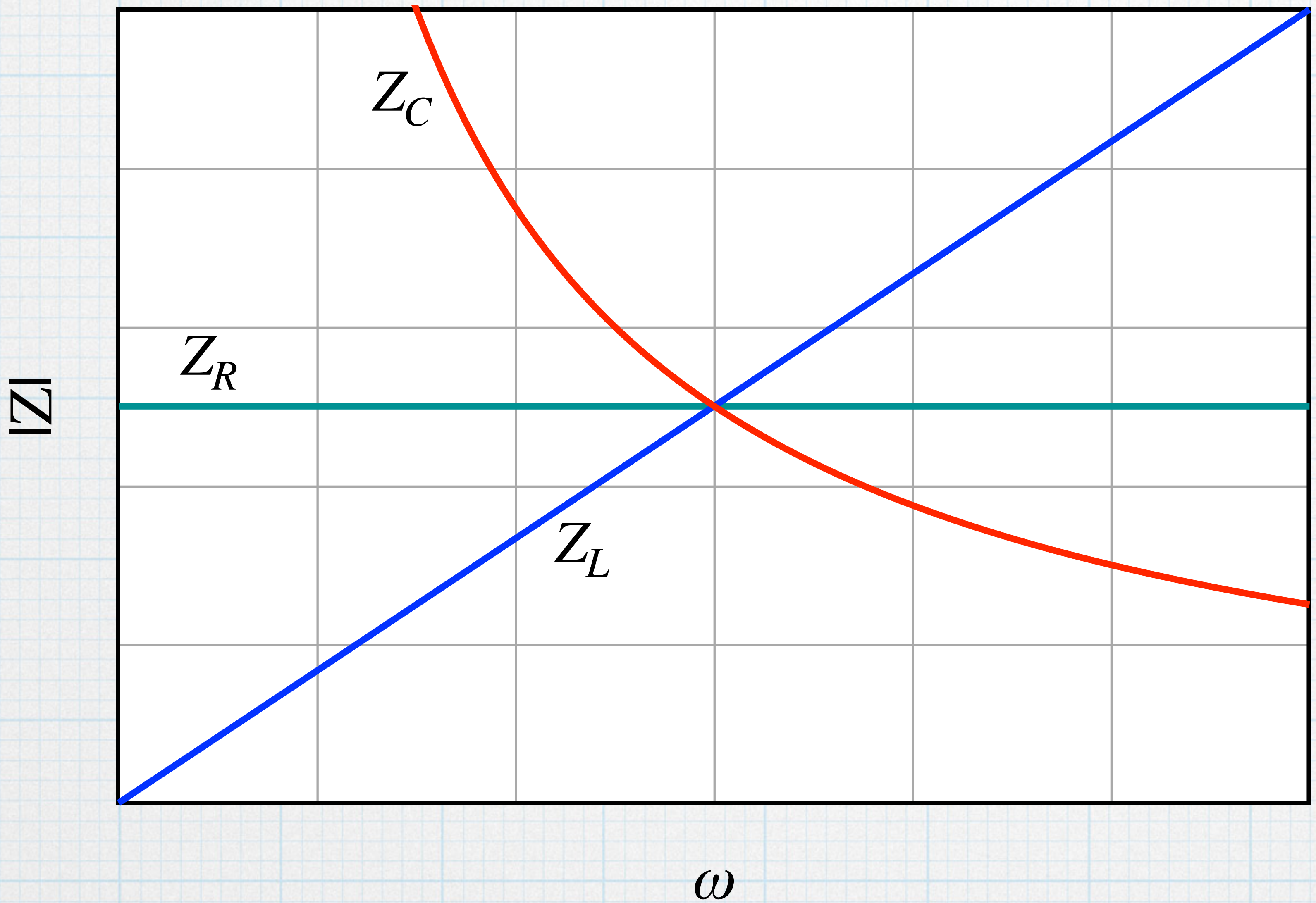
$$Z_C = \frac{1}{j\omega C} = -j\left(\frac{1}{\omega C}\right)$$

One-line proof: $\frac{1}{j} = \frac{1}{e^{j90^\circ}} = e^{-j90^\circ} = -j$

So for circuits that have inductors and capacitors, the impedances may tend to cancel each other. It is possible that they may cancel exactly ($Z_L = -Z_C$) — a situation that we call *resonance*. This “fight” between inductor impedance and capacitive impedance has important implications and applications. We will study some of these later in 201 and in EE 230.

Equally important is the frequency dependence of the two impedances. The inductor impedance increases proportionally with increasing frequency, the capacitor impedance decreases inversely with increasing frequency — exact opposites.

Frequency dependence of the magnitudes of the three impedances.



Resistors are frequency independent. Capacitors and inductors change behavior as the frequency changes.

Having all three impedances meet at a single frequency, as shown in this graph would be unusual. But we could certainly force that to happen by making appropriate choices for R, L, C, and ω!

$$Z_L = j\omega L$$

$$Z_C = \frac{1}{j\omega C}$$

At very low frequencies ($\omega \rightarrow 0$, which is just another way of describing DC),

$$|Z_L| \rightarrow 0 \text{ and } |Z_C| \rightarrow \infty.$$

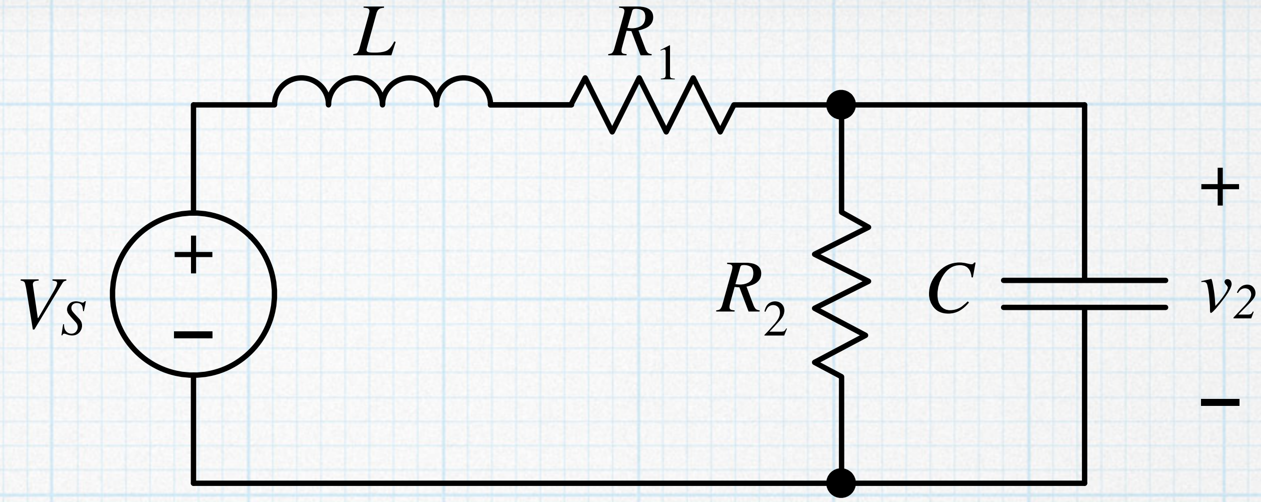
At DC, the inductor becomes a short circuit and the capacitor becomes an open circuit. This is not a surprise — this is exactly how we introduced the inductor and capacitor. At DC, an inductor is a fancy short circuit and a capacitor is fancy open circuit.

But at very high frequencies ($\omega \rightarrow \infty$), the situation is quite different:

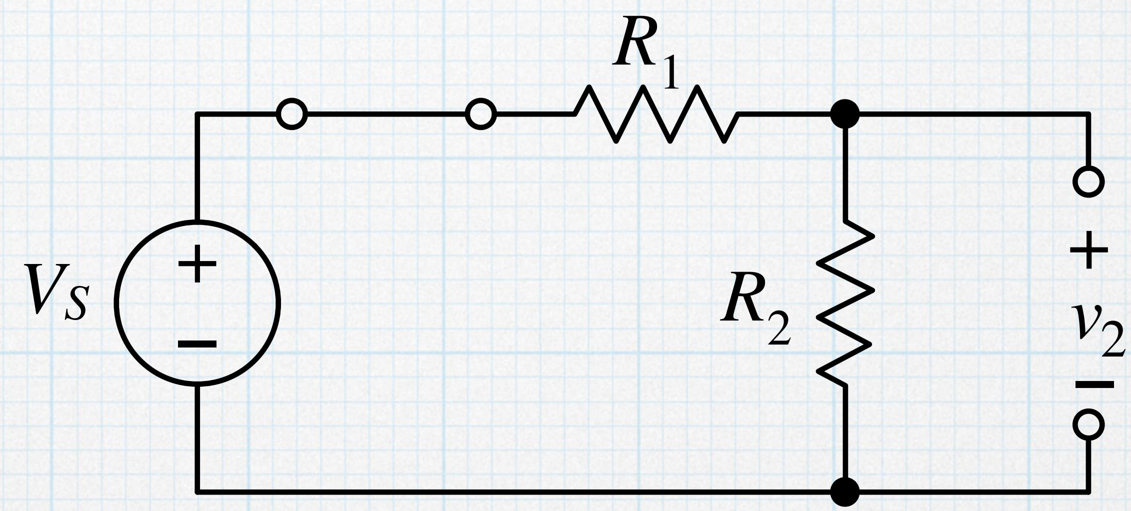
$$|Z_L| \rightarrow \infty \text{ and } |Z_C| \rightarrow 0!$$

The inductor and capacitor have completely switched roles. This is quite unexpected. Obviously, the ways that impedance changes with frequency will have a big impact on how a circuit behaves at different frequencies.

Consider this *RLC* circuit. V_S is a sinusoid.

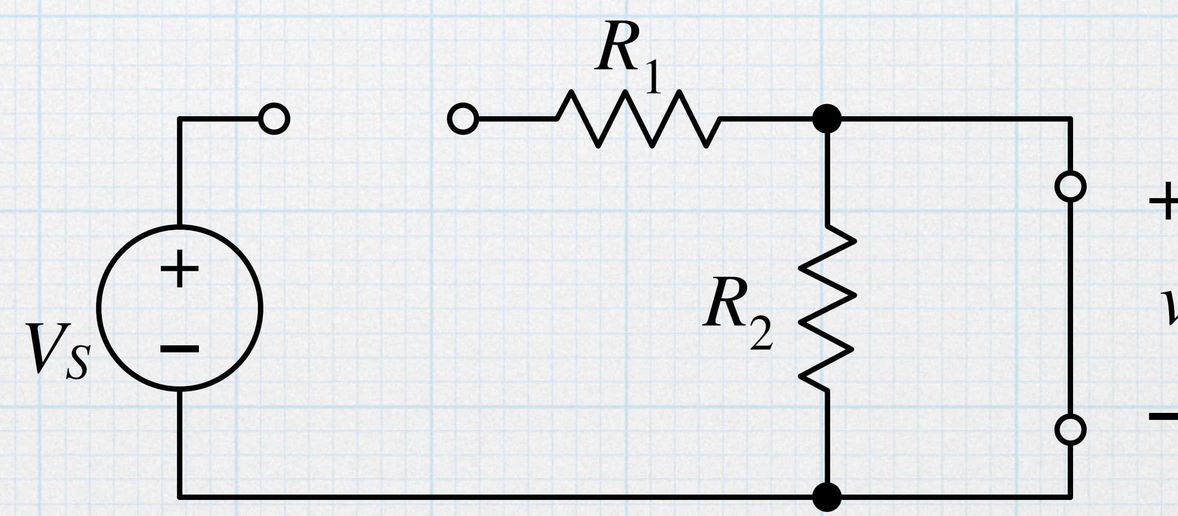


Suppose the frequency is very low, $\omega \to 0$. (i.e. DC.) The inductor behaves like a short and the capacitor like an open.



$$v_2 = \frac{R_2}{R_1 + R_2} V_S$$

Now let the frequency be very high, $\omega \to \infty$. The inductor behaves like a open and the capacitor like a short!



$$v_2 = 0!$$

Using the frequency dependence of the impedances is an important part of manipulating and processing signals. (EE 230, EE 224)

Using impedances to analyze AC circuits

Impedances give us the opportunity to re-use the circuit analysis techniques that we learned at the beginning of 201. Recall that we combined Kirchoff's Laws together with Ohm's Law to solve many different resistive DC circuits using a variety of methods. Impedances are quantities that relate voltages and currents in exactly same way as resistors. This implies that we could use impedances together with Kirchoff's Laws to solve AC circuits, using the same methods (dividers, node voltage, mesh current, etc.) that we learned earlier. Except that now the calculations will be done using complex numbers. We take on the burden of complex math so that we can return to our familiar circuit analysis methods and stop solving messy differential equations.

So our AC analysis approach will be to first convert the AC circuit to its complex equivalent. (Sources become complex sinusoids and components become impedances.) Then we ask ourselves the question: "If the impedances were all resistors, what method would we use to find DC voltages and currents?" Then we apply that method using impedances instead of resistors. There will be complex math involved — maybe a little or maybe a lot. When we get through that, the results will be complex voltages and currents, which tell us the amplitudes and phase angles of the sinusoids.

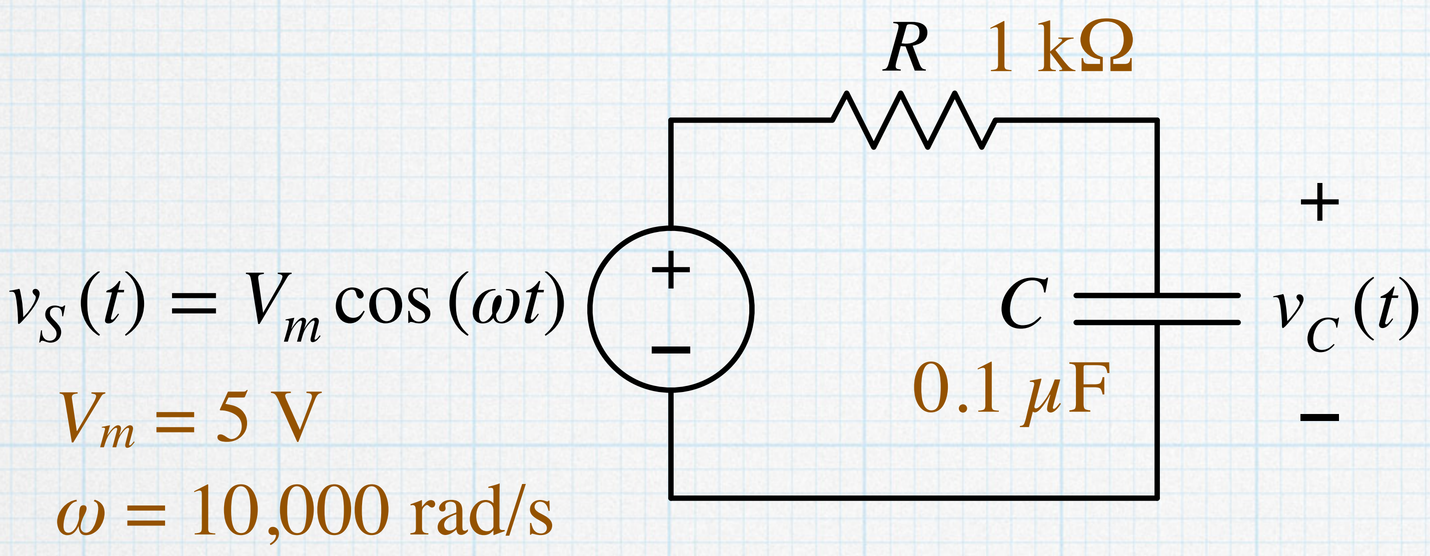
The procedure for solving AC circuits using impedances is:

1. Convert the sources to exponential sinusoids. If there is only one source, we can choose its phase to be zero. If there are multiple sources, we will need to keep track of any phase differences between them.
2. Convert the components to impedances.
3. Use the usual collection of tools (dividers, node voltage, mesh current, etc.) to set up equations relating voltages and currents in the circuit.
4. Grind through the complex algebra to find the complex values for the voltages and currents. (This is the worst part.)
5. When finished, express the complex voltages and currents in magnitude and phase form. The magnitude is amplitude of the oscillation and the phase is the shift relative to the source. These are the features that we would observe if the we built the circuit in the lab and used an oscilloscope to view the sinusoidal wave forms.

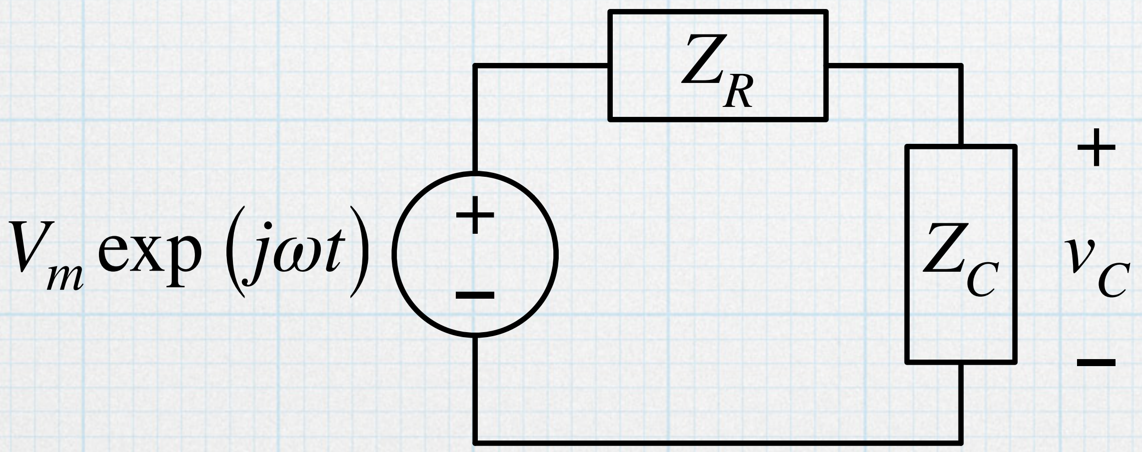
Following are many examples, starting with a slew of dividers.

Example 1

Use AC analysis to find the sinusoidal form of the resistor voltage below.



Transform to the complex version of the circuit.



$$Z_R = R$$

$$Z_C = \frac{1}{j\omega C}$$

In viewing the complex circuit and thinking about how to approach it if the impedances were all resistors, it seems that using a voltage divider would be a reasonable approach.

$$v_C = \frac{Z_C}{Z_R + Z_C}\left[V_m \exp(j\omega t)\right]$$

Working out the value for the voltage divider ratio:

$$\frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}$$

$$= \frac{-j\left(\frac{1}{\omega C}\right)}{R - j\left(\frac{1}{\omega C}\right)}$$

Mind the negatives. ($1/j = -j$!)

$$= \frac{-j1000\,\Omega}{1000\,\Omega - j1000\,\Omega}$$

Plug in the numbers.

$$= \frac{(1000\,\Omega)\,e^{-j90^\circ}}{(1414\,\Omega)\,e^{-j45^\circ}}$$

Convert to magnitude/phase.

$$= (0.707)\,e^{-j45^\circ}$$

Finish.

Then the resistor voltage is

$$v_R = \left[(0.8)\,e^{-j36.9^\circ}\right]\left[\left(5\text{ V}\right)\exp\left(j\omega t\right)\right] = (4\text{ V})\,e^{j\left(\omega t - 36.9^\circ\right)}$$

$$v_R = (4\text{ V})\, e^{j(\omega t - 36.9^\circ)}$$

The result is a sinusoid with amplitude of 4 V, shifted in phase by –36.9° from the source.

Plugging in numbers early in the calculation is OK, but sometimes the complex math is a little easier if we do a bit more algebra with symbols before switching to numbers. We might also gain some insight what is going on in the circuit. Go back to the voltage divider ratio:

$$\frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}$$

$$= \frac{1}{1 + j\omega RC}$$

Multiply top and bottom by $j\omega C$.

$$= \frac{1}{1 + j1}$$

Now plug in the numbers

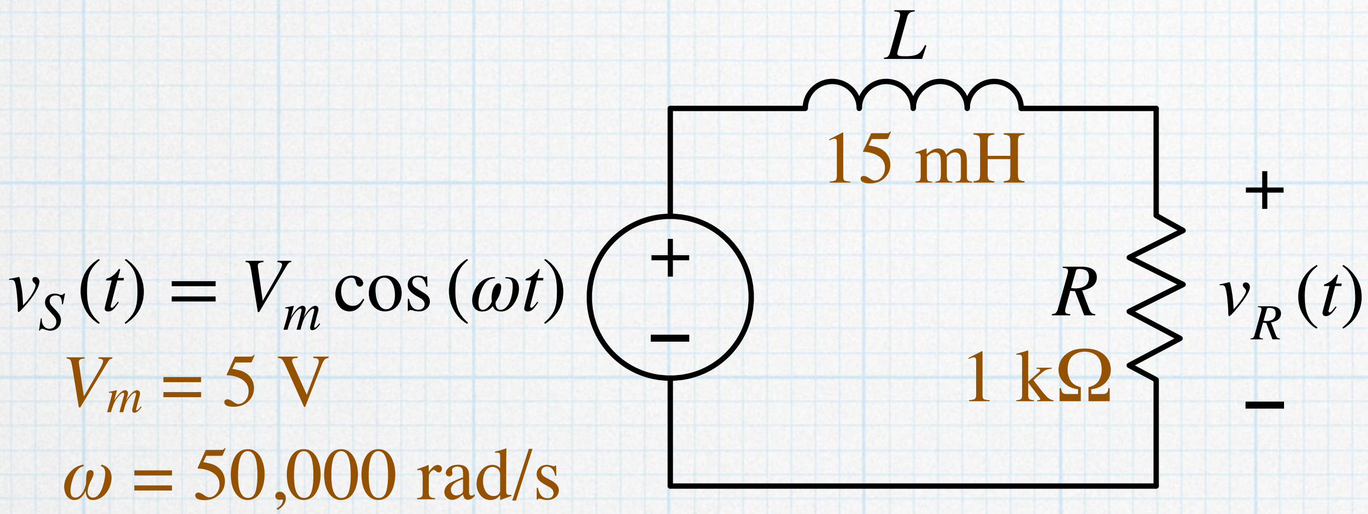
$$= \frac{1}{1.414 \cdot e^{j45^\circ}} = 0.707 \cdot e^{-j45^\circ}$$

And we get the same ratio.

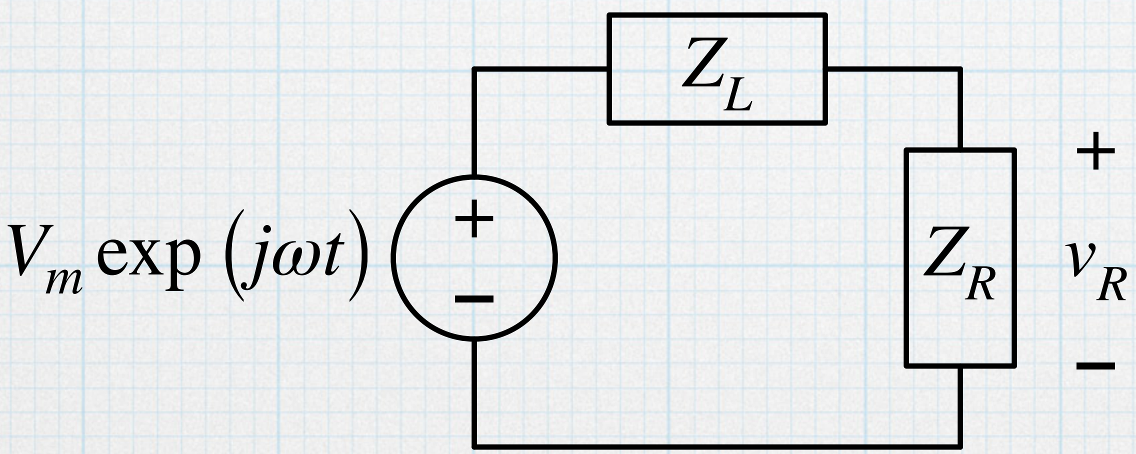
In this case, there is not a huge difference in the math process, but we do see a simplification that comes from using dimensionless quantities. (Note that ωRC is dimensionless — check the units.)

Example 2

Use AC analysis to find the sinusoidal form of the resistor voltage below.



Transform to the complex version of the circuit.



$Z_L = j\omega L$

$Z_R = R$

Again, using a voltage divider seems like a reasonable approach.

$$v_L = \frac{Z_R}{Z_R + Z_L}\left[V_m \exp(j\omega t)\right]$$

Working out the value for the voltage divider ratio:

$$\frac{Z_R}{Z_R + Z_L} = \frac{R}{R + j\omega L}$$

$$= \frac{1000\,\Omega}{1000\,\Omega + j750\,\Omega}$$ Plug in the numbers.

$$= \frac{1000\,\Omega}{\left(1250\,\Omega\right) e^{+j36.9^\circ}}$$ Convert to magnitude/phase.

$$= (0.8)\, e^{-j36.9^\circ}$$ Finish.

Then the resistor voltage is

$$v_R = \left[(0.8)\, e^{-j36.9^\circ}\right] \left[\left(5\text{ V}\right) \exp\left(j\omega t\right)\right]$$

$$v_R = (4\text{ V})\, e^{j\left(\omega t - 36.9^\circ\right)}$$

A sinusoid with amplitude of 4 V, shifted in phase by –36.9° from the source.

Alternatively, we can use a math trick similar to Example 1:

$$\frac{Z_R}{Z_R + Z_L} = \frac{R}{R + j\omega L}$$

$$= \frac{1}{1 + j\frac{\omega L}{R}}$$ Divide top and bottom by R.

$$= \frac{1}{1 + j0.75}$$ Plug in the numbers

$$= \frac{1}{\left(1.25\right) e^{+j36.9°}}$$ Convert to magnitude/phase.

$$= (0.8)\, e^{-j36.9°}$$ Finish.

The final result would be the same. Note that the quantity $\omega L / R$ is dimensionless. (Check it.)

Time to drop the $e^{j\omega t}$ factor

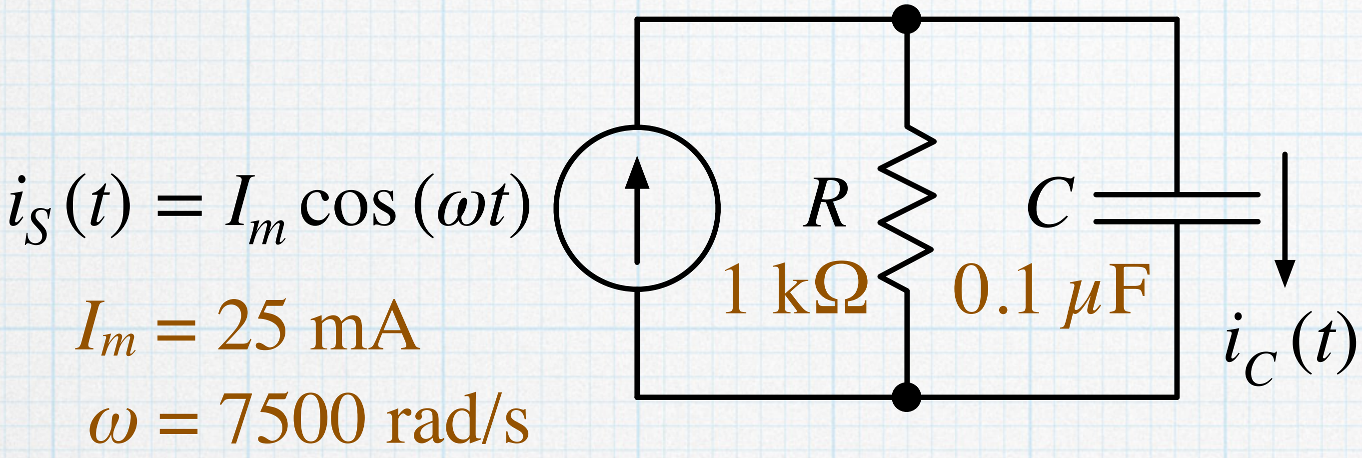
At the end of each of the previous examples, the answers were expressed as complex sinusoids, like $v_R = (4\,\mathrm{V})\, e^{j\left(\omega t - 36.9°\right)}$. It should fairly obvious that *every* voltage and current in the circuit will have the factor $e^{j\omega t}$ built into it. The $e^{j\omega t}$ tells us that the quantity is oscillating with the angular frequency ω. Of course, we *know* that it will be oscillating at that frequency — the driving source sets up all the voltages and currents in the circuit. The source sets the pace and everything else must follow.

Since we know that all quantities will be multiplied by $e^{j\omega t}$, there is really no need to include it at each step. All we really need is the magnitude and phase of the voltage or current. Writing the above answer as $v_R = (4\,\mathrm{V})\, e^{-j36.9°}$ tells us everything we need to know. So from here on, we acknowledge that we know that everything is oscillating in the same manner, and we agree that we don't need to attach $e^{j\omega t}$ everywhere. We know that is there implicitly, but we don't need to write it explicitly.

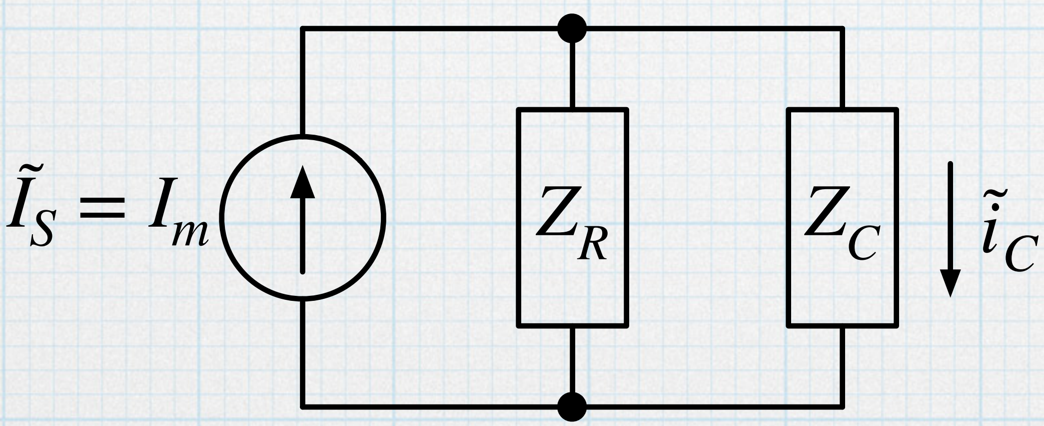
So a voltage source can expressed more simply: $V_m e^{j(\omega t - 0°)} \rightarrow V_m e^{j0°} \rightarrow V_m$. To help us remember that the quantities in the circuit are complex numbers representing sinusoids, we will add a little hat (tilde) over voltages or currents: $\tilde{V}_S = 10\,\mathrm{V}$ or $\tilde{v}_R = (4\,\mathrm{V})\, e^{-j36.9°}$. (Looks like a tiny sine wave.)

Example 3

Use AC analysis to find the capacitor current in the circuit below.



Transform to the complex version of the circuit. (Use the new notation.)



$$Z_R = R$$

$$Z_C = \frac{1}{j\omega C}$$

The theme of the day is dividers — use a current divider.

$$\tilde{i}_C = \frac{\frac{1}{Z_C}}{\frac{1}{Z_R} + \frac{1}{Z_C}} \tilde{I}_S$$

Working out the details for the current divider ratio:

$$\frac{\frac{1}{Z_C}}{\frac{1}{Z_R}+\frac{1}{Z_C}} = \frac{j\omega C}{\frac{1}{R}+j\omega C}$$

$$= \frac{j0.75\,\mathrm{mS}}{1\,\mathrm{mS}+j0.75\,\mathrm{mS}}$$ Plug in the numbers. ($\Omega^{-1} = S$)

$$= \frac{\left(0.75\,\mathrm{mS}\right)e^{+j90^\circ}}{\left(1.25\,\mathrm{mS}\right)e^{+j36.9^\circ}}$$ Convert to magnitude/phase.

$$= \left(0.6\right)e^{+j53.1^\circ}$$ Finish.

Then the complex capacitor current is

$$\tilde{i}_C = \left[\left(0.6\right)e^{j53.1^\circ}\right]\left(15\,\mathrm{mA}\right) = \left(25\,\mathrm{mA}\right)e^{j53.1^\circ}$$

The capacitor current will be a sinusoid oscillating with angular frequency ω = 7500 rad/s, having an amplitude of 15 mA and shifted in phase by 53.1° from the source.

Continuing with the math tricks,

$$\frac{\frac{1}{Z_C}}{\frac{1}{Z_R}+\frac{1}{Z_C}} = \frac{j\omega C}{\frac{1}{R}+j\omega C}$$

$$= \frac{1}{1+\left(\frac{1}{j\omega RC}\right)}$$ Divide top and bottom by $j\omega C$.

$$= \frac{1}{1-j1.333}$$ Plug in the numbers

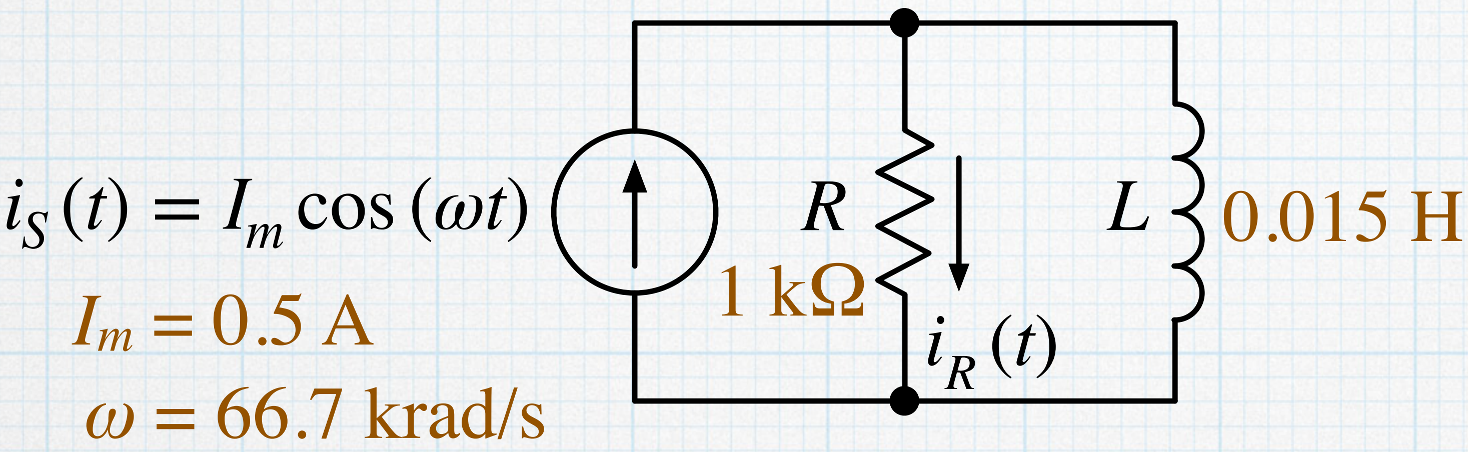
$$= \frac{1}{(1.667)\, e^{-j53.1^\circ}}$$ Convert to magnitude/phase.

$$= (0.6)\, e^{j53.1^\circ}$$ Finish.

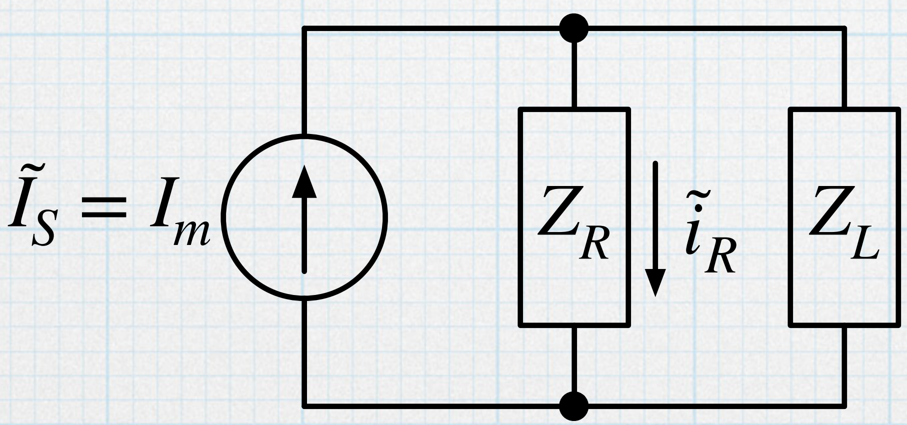
We should no longer be surprised that the result is the same.

Example 4

Use AC analysis to find the resistor current in the circuit below.



Transform to the complex version of the circuit.



$$Z_R = R$$
$$Z_L = j\omega L$$

Ho Hum. Another day, another divider.

$$\tilde{i}_R = \frac{\frac{1}{Z_R}}{\frac{1}{Z_R} + \frac{1}{Z_L}} \tilde{I}_S$$

Looking at the current divider ratio:

$$\frac{\frac{1}{Z_R}}{\frac{1}{Z_R}+\frac{1}{Z_L}} = \frac{\frac{1}{R}}{\frac{1}{R}+\frac{1}{j\omega L}}$$

$$= \frac{1\text{ mS}}{1\text{ mS} - j1\text{ mS}}$$ Plug in the numbers. $(1/j = -j\ !)$

$$= \frac{1\text{ mS}}{(1.414\text{ mS})\, e^{-j45^\circ}}$$ Convert to magnitude/phase.

$$= (0.707)\, e^{+j45^\circ}$$ And done.

Then the complex resistor current is

$$\tilde{i}_R = \left[(0.707)\, e^{j45^\circ}\right] \left(0.5\text{ A}\right) = \left(0.354\text{ A}\right) e^{j45^\circ}$$

The resistor current is a sinusoid oscillating with angular frequency $\omega = 66{,}700$ rad/s, having an amplitude of 0.354 A and shifted in phase by 45° from the source.

More math trickery,

$$\frac{\frac{1}{Z_R}}{\frac{1}{Z_R} + \frac{1}{Z_L}} = \frac{\frac{1}{R}}{\frac{1}{R} + \frac{1}{j\omega L}}$$

$$= \frac{1}{1 + \left(\frac{R}{j\omega L}\right)}$$ Multiply top and bottom by R.

$$= \frac{1}{1 - j1}$$ Plug in the numbers.

$$= \frac{1.414}{\left(1.667\right) e^{-j45^\circ}}$$ Convert to magnitude/phase.

$$= (0.707)\, e^{j45^\circ}$$ Finish.

Yup.

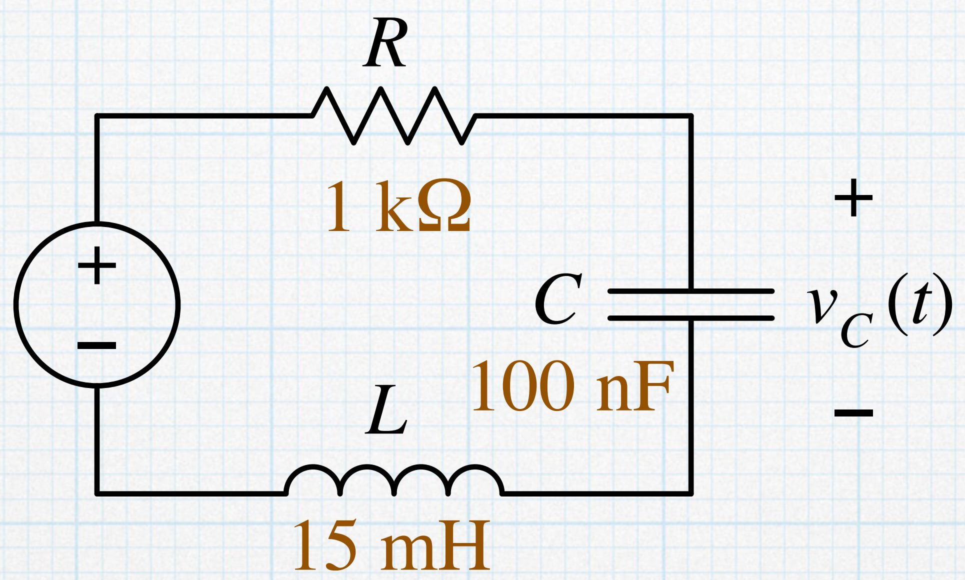
Example 5

It's more fun when there are resistors, capacitors, *and* inductors all together. Find the important sinusoid information about the capacitor voltage in the circuit.

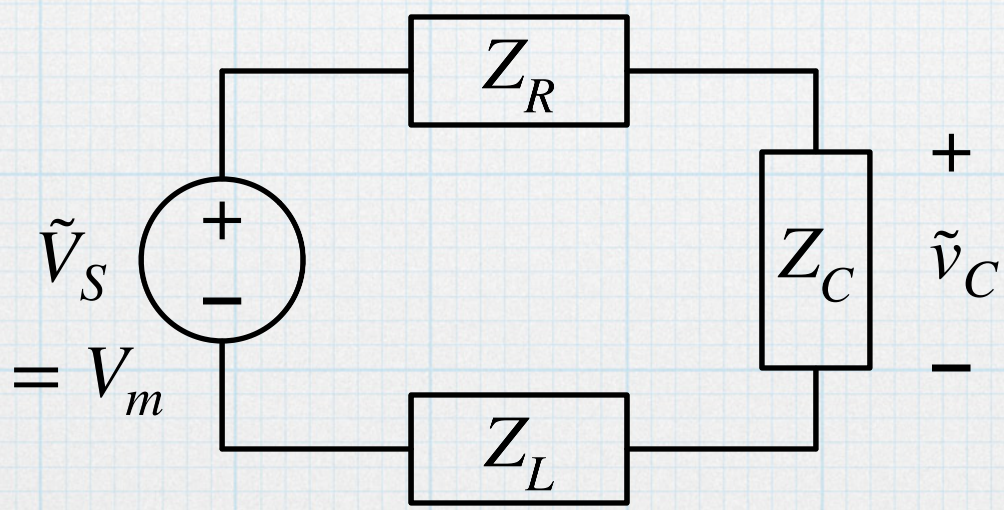
$v_S(t) = V_m \cos(\omega t)$

$V_m = 5$ V

$\omega = 40{,}000$ rad/s



Transform to the complex version of the circuit.



$Z_R = R$

$Z_C = \dfrac{1}{j\omega C}$

$Z_L = j\omega L$

It's hard to avoid voltage dividers.

$$\tilde{v}_C = \frac{Z_C}{Z_R + Z_C + Z_L}\tilde{V}_S$$

$$\frac{Z_C}{Z_R + Z_C + Z_L} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C} + j\omega L}$$

$$= \frac{-j250\,\Omega}{1000\,\Omega - j250\,\Omega + j600\,\Omega}$$ Plug in the numbers.

$$= \frac{-j250\,\Omega}{1000\,\Omega + j350\,\Omega}$$ Combine.

$$= \frac{(250\,\Omega)\, e^{-j90^\circ}}{(1060\,\Omega)\, e^{+j19.3^\circ}}$$ Convert.

$$= (0.236)\, e^{-j109.3^\circ}$$ Finish.

Then the complex capacitor voltage is

$$\tilde{v}_R = \left[(0.236)\, e^{-j109.3^\circ}\right](5\text{ V}) = (1.18\text{ V})\, e^{-j109.3^\circ}$$

A sinusoid with amplitude of 1.18 V, shifted in phase by –109.3° from the source.

There are some manipulations we can apply here as well.

$$\frac{Z_C}{Z_R + Z_C + Z_L} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C} + j\omega L}$$

$$= \frac{1}{j\omega RC + 1 - \omega^2 LC}$$ Multiply top/bottom by $j\omega C$.

$$= \frac{1}{\left(1 - \omega^2 LC\right) + j\left(\omega RC\right)}$$ Re-arrange.

$$= \frac{1}{-1.4 + j4}$$ Plug in the numbers

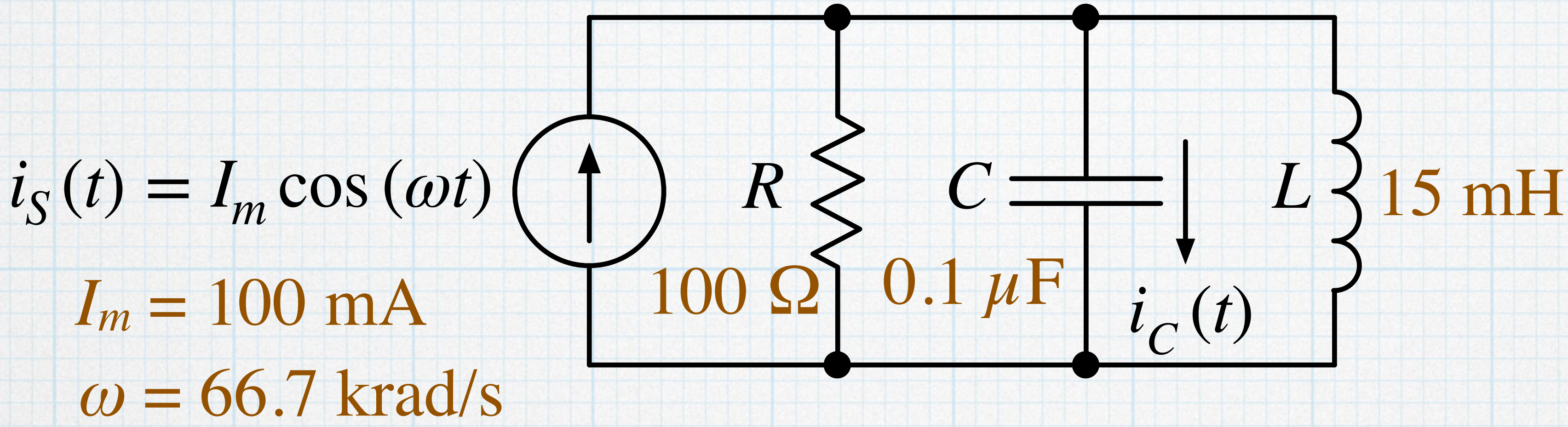
$$= \frac{1}{(4.24)\, e^{+j109.3^\circ}}$$ Convert.

$$= \left(0.236\right) e^{-j109.3^\circ}$$ Finish.

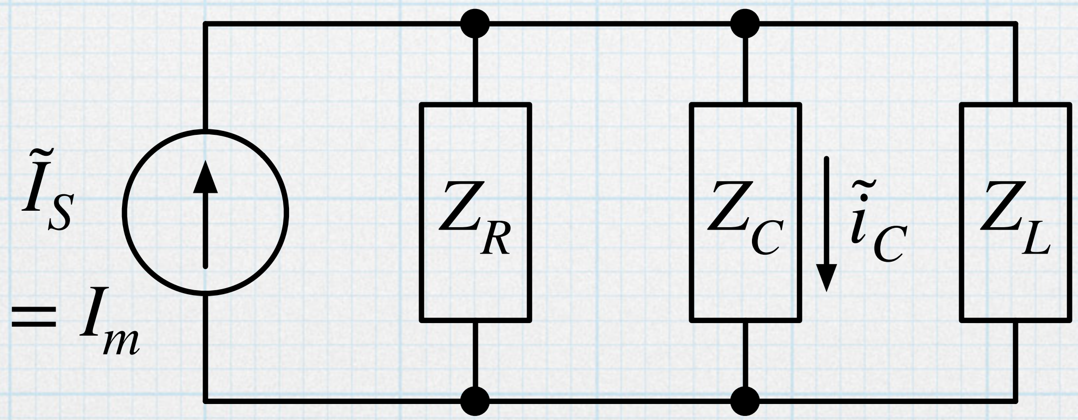
Note yet another dimensionless quantity: $\omega^2 LC$. (Check this, too.)

Example 6

We should do an *RLC* current divider — it wouldn't be right to ignore it. Find the capacitor current.



Transform to the complex version of the circuit.



$$Z_R = R$$

$$Z_C = \frac{1}{j\omega C}$$

$$Z_L = j\omega L$$

We know how this goes.

$$\tilde{i}_C = \frac{\frac{1}{Z_C}}{\frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L}} \tilde{I}_S$$

$$\frac{\frac{1}{Z_C}}{\frac{1}{Z_R}+\frac{1}{Z_C}+\frac{1}{Z_L}} = \frac{j\omega C}{\frac{1}{R}+j\omega C+\frac{1}{j\omega L}}$$

$$= \frac{j6.67\,\text{mS}}{10\,\text{mS}+j6.67\,\text{mS}-j1\,\text{mS}}$$ Plug in the numbers.

$$= \frac{j6.67\,\text{mS}}{10\,\text{mS}+j5.67\,\text{mS}}$$ Combine.

$$= \frac{\left(6.67\,\text{mS}\right)e^{+j90^\circ}}{\left(11.5\,\text{mS}\right)e^{+j29.6^\circ}}$$ Convert.

$$= \left(0.58\right)e^{+j60.4^\circ}$$ Finish.

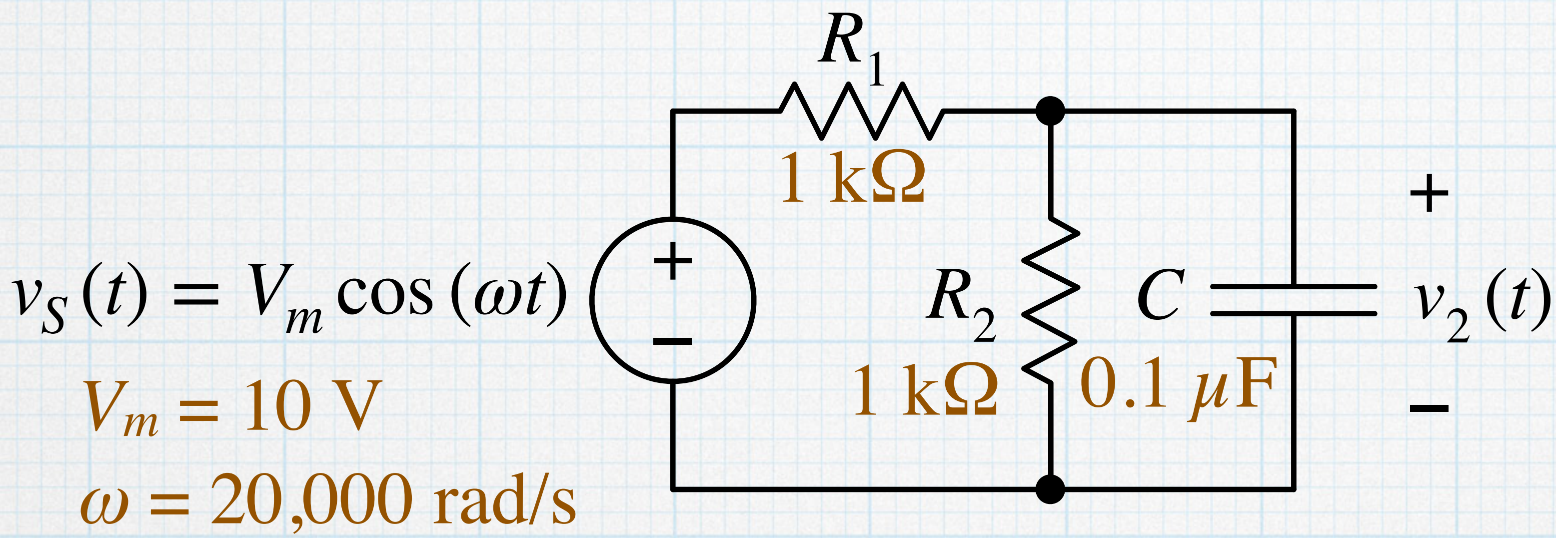
Then the complex capacitor current is

$$\tilde{i}_C = \left[\left(0.58\right)e^{j60.4^\circ}\right](100\,\text{mA}) = \left(58\,\text{mA}\right)e^{j60.4^\circ}$$

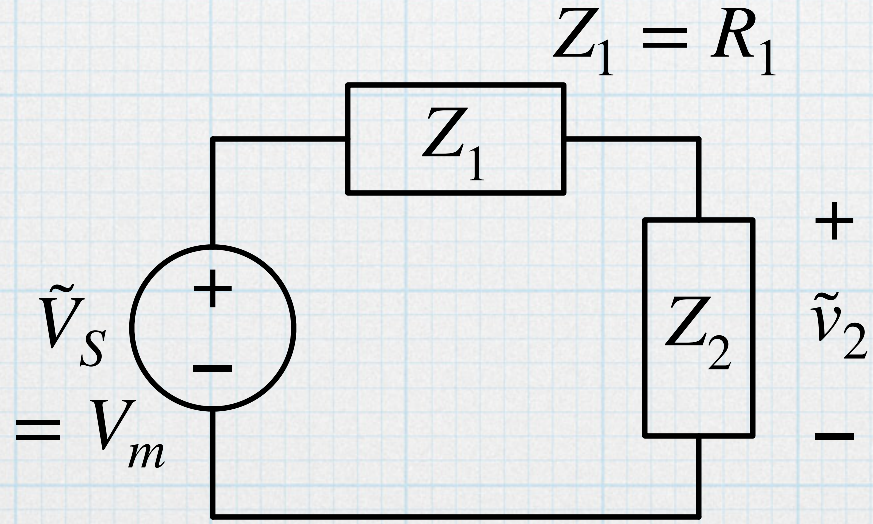
We know what this means. We will skip the alternate math here, but you should try a math trick for yourself. (Multiply by $j\omega L$ or divide by $j\omega C$.)

Example 7

Let's try this circuit— it looks a bit like the *RC* circuit from example 1. Find the voltage across the parallel combination of R_2 and C.



In fact, this is yet another simple voltage divider, if we treat the R_2-C combination as a single impedance.



$$Z_2 = Z_{R2} \| Z_C = R_2 \| \left(\frac{1}{j\omega C}\right)$$

$$= \frac{(R_2)\left(\frac{1}{j\omega C}\right)}{R_2 + \frac{1}{j\omega C}}$$

$$= \frac{R_2}{1 + j\omega R_2 C}$$

$$\tilde{v}_2 = \frac{Z_2}{Z_1 + Z_2}\tilde{V}_S$$

Inserting the impedances into the voltage divider expression :

$$\frac{Z_2}{Z_1 + Z_2} = \frac{\dfrac{R_2}{1 + j\omega R_2 C}}{R_1 + \dfrac{R_2}{1 + j\omega R_2 C}}$$

Yikes! That looks messy. At this point in previous examples, we were able to substitute values directly and do the complex calculations without too much hassle. In this case, if we substitute values now, there will be many conversions back and forth between real-imaginary and magnitude-phase — the math will be quite tedious and prone to errors. In the earlier examples, we also showed that a bit of algebraic manipulation could make the math a bit cleaner, although the extra steps were not really necessary in those simpler problems. However, in this example, some algebraic manipulation is not just advisable, it is probably necessary in order to make the ensuing math tenable. To simplify this expression, we can multiply top and bottom by $1 + j\omega R_2 C$:

$$\frac{Z_2}{Z_1 + Z_2} = \frac{R_2}{R_1 + R_2 + j\omega R_1 R_2 C}$$

That's nicer.

$$= \frac{1000\,\Omega}{2000\,\Omega + j2000\,\Omega}$$

Inserting values. Not bad at all.

Continuing with voltage divider ratio:

$$\frac{Z_2}{Z_1 + Z_2} = \frac{1000\,\Omega}{2000\,\Omega + j2000}$$

$$= \frac{1000\,\Omega}{(2828\,\Omega)\,e^{+j45^\circ}}$$ Convert to magnitude/phase.

$$= \left(0.354\right)e^{-j45^\circ}$$ Finish.

Then the complex voltage across the parallel combination is

$$\tilde{v}_2 = \left[\left(0.354\right)e^{-j45^\circ}\right](10\,\mathrm{V}) = \left(3.54\,\mathrm{V}\right)e^{-j45^\circ}$$

There is one additional algebra step we might have tried before inserting values. If, after the first simplification, we *divide* top and bottom by R_1+R_2, the divider expression becomes:

$$\frac{\dfrac{R_2}{R_1 + R_2}}{1 + j\omega\left(\dfrac{R_1 R_2}{R_1 + R_2}\right)C} = \frac{0.5}{1 + j1} = \frac{0.5}{\sqrt{2}e^{j45^\circ}} = 0.354e^{-j45^\circ}$$

The extra step is not essential, but it does offer some insights.

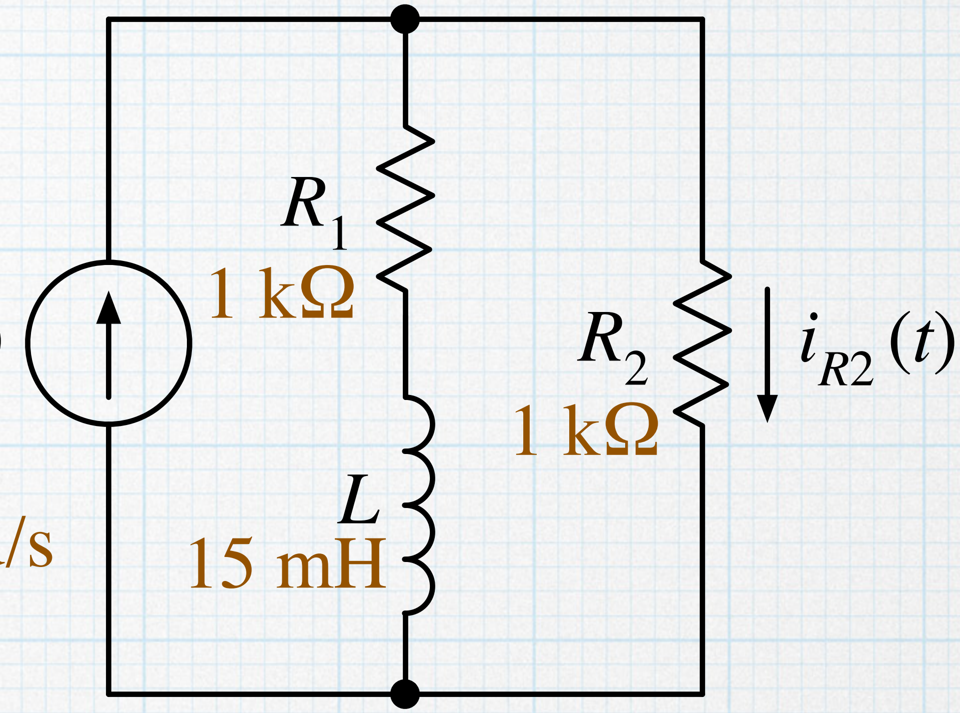
Example 8

One more divider — find the key features of the sinusoidal current i_{R2}.

$$i_S(t) = I_m \cos(\omega t)$$

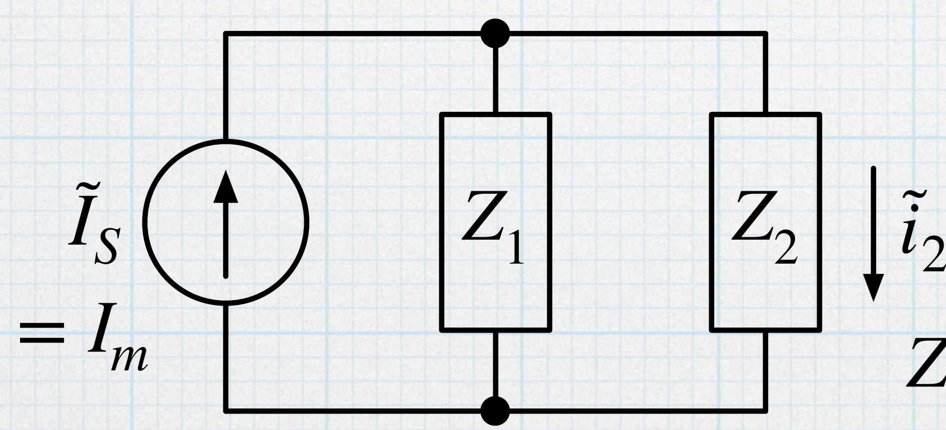
$I_m = 50$ mA

$\omega = 50{,}000$ rad/s



Again, we have a simple current divider if we treat the R_1-L series combination as a single impedance.

$$Z_1 = Z_{R1} + Z_L = R_1 + j\omega L$$



$Z_2 = R_2$

$$\tilde{i}_2 = \frac{\frac{1}{Z_2}}{\frac{1}{Z_1} + \frac{1}{Z_2}} \tilde{I}_S$$

Inserting the impedances into the voltage divider expression:

$$\frac{\frac{1}{Z_2}}{\frac{1}{Z_1}+\frac{1}{Z_2}} = \frac{\frac{1}{R_2}}{\frac{1}{R_1 + j\omega L}+\frac{1}{R_2}}$$

This could use some tidying up before inserting numbers. Multiply top and bottom by $R_1 + j\omega L$:

$$= \frac{\frac{R_1}{R_2} + j\left(\frac{\omega L}{R_2}\right)}{1 + \frac{R_1}{R_2} + j\left(\frac{\omega L}{R_2}\right)}$$

That's better.

$$= \frac{1 + j0.75}{2 + j0.75}$$

Inserting values — nice and clean.

$$= \frac{(1.25)\, e^{j36.9^\circ}}{(2.136)\, e^{j20.6^\circ}} = (0.585)\, e^{j16.3^\circ}$$

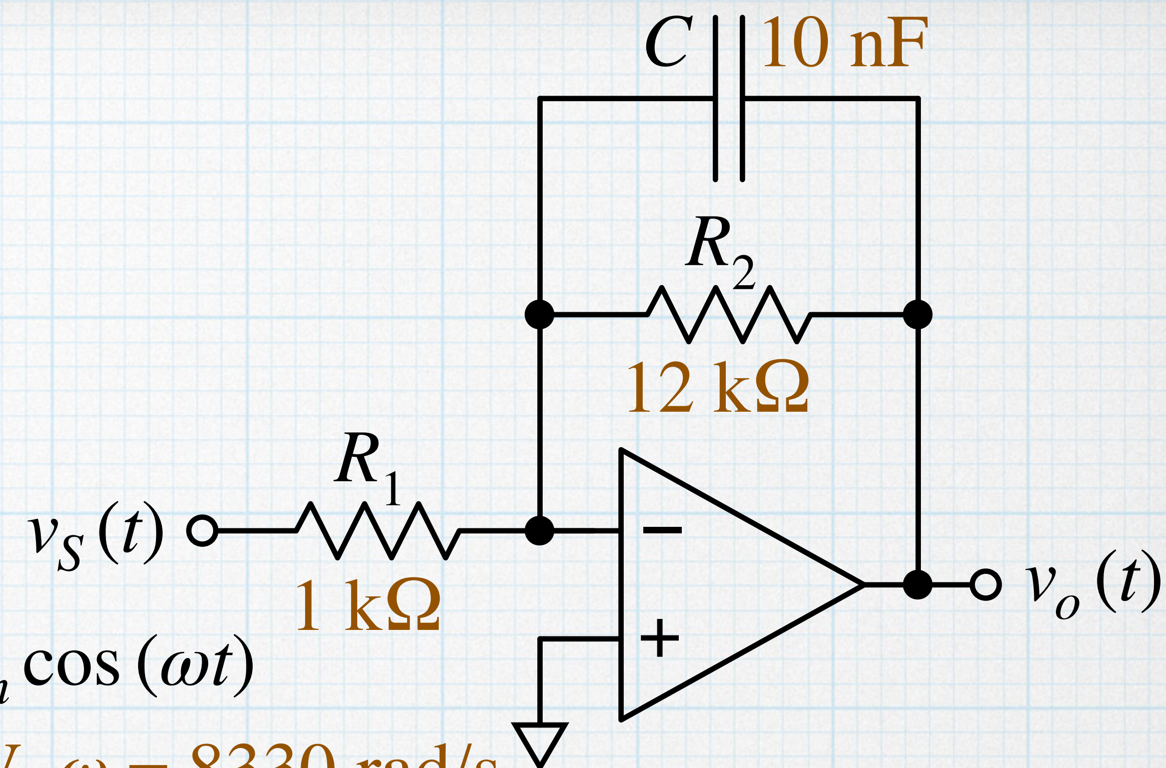
Then the complex current through R_2 is

$$\tilde{i}_2 = \left[(0.585)\, e^{j16.3^\circ}\right](50\,\text{mA}) = (29.25\,\text{mA})\, e^{j16.3^\circ}$$

Example 9

Let's try an op amp. Find the important information (magnitude, phase) for the sinusoidal output voltage.

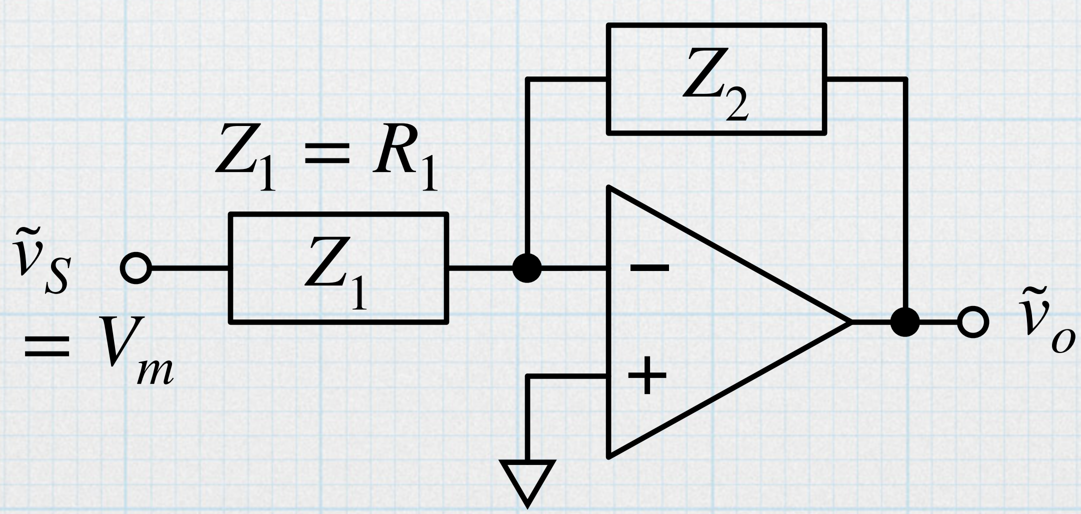
(Not a divider! Woo Hoo!)



$v_S(t) = V_m \cos(\omega t)$

$V_m = 0.5$ V $\omega = 8330$ rad/s

How do ops amp work in a complex circuit? Just like they did before. The rules are unchanged: no current flows into the input, $\tilde{i}_+ = \tilde{i}_- = 0$, and when there is a negative feedback loop, $\tilde{v}_+ = \tilde{v}_-$.
If we treat the resistor/capacitor parallel combination as a single impedance, the complex version of the circuit has the form of a simple inverting amp.



$$Z_2 = Z_{R2} \| Z_C$$

$$= \frac{R_2}{1 + j\omega R_2 C}$$

(As seen previously.)

$$G = \frac{\tilde{v}_o}{\tilde{v}_s} = -\frac{Z_2}{Z_1}$$

$$G = -\frac{Z_2}{Z_1}$$

$$= -\frac{\frac{R_2}{1 + j\omega R_2 C}}{R_1}$$

$$= \frac{-\frac{R_2}{R_1}}{1 + j\omega R_2 C}$$

With a tiny bit of re-arrangement:

$$= \frac{-12}{1 + j1}$$

Inserting values

$$= \frac{(12)\, e^{j180^\circ}}{(1.414)\, e^{j45^\circ}}$$

Transform to polar. Watch the negative sign!

$$= (8.49)\, e^{j135^\circ}$$

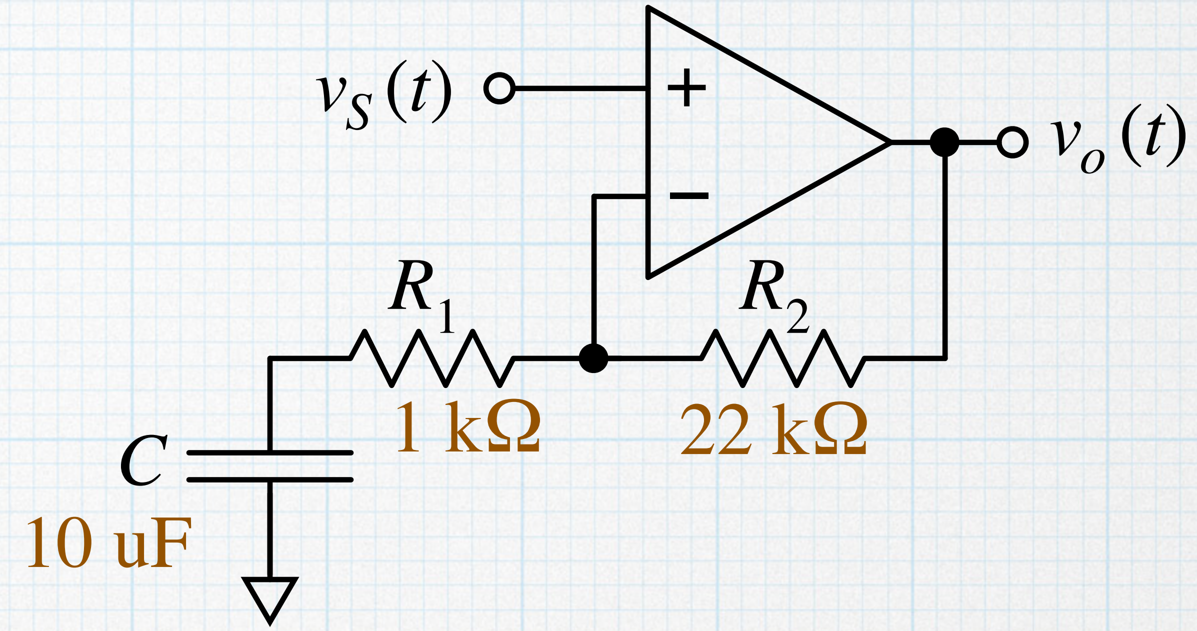
Then the complex output voltage is

$$\tilde{v}_o = G \cdot \tilde{v}_S = \left[(8.49)\, e^{j135^\circ}\right] \left(0.5\text{ V}\right) = (4.24\text{ V})\, e^{j135^\circ}$$

Op amps are so easy.

Example 10

Op amps are fun. Let's do one more. Find the complex output voltage for the non-inverting amp at right.

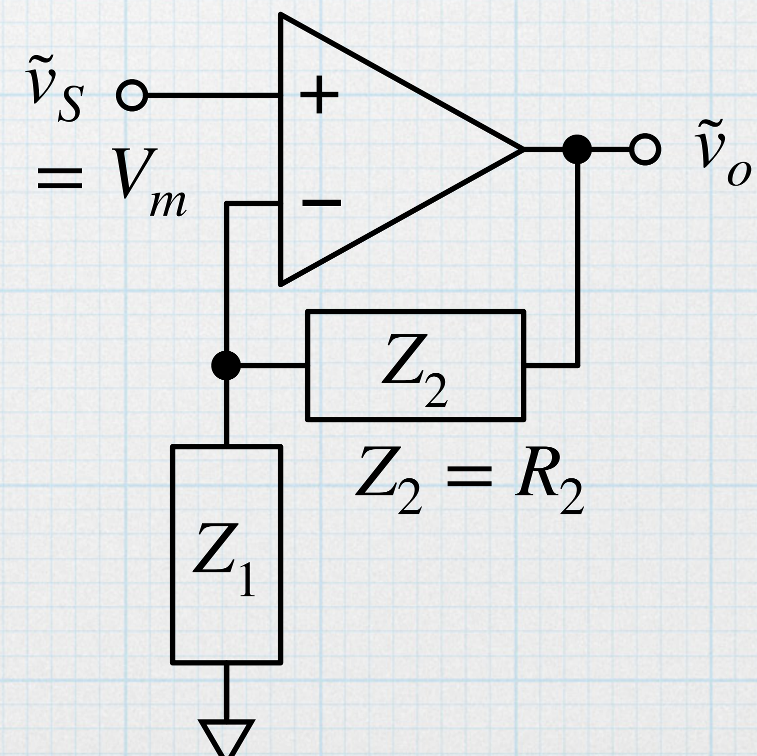


$$v_S(t) = V_m \cos(\omega t)$$

$V_m = 0.5$ V $\omega = 100$ rad/s

It may not be immediately obvious that this is a non-inverting amp, but if we treat the R_1-C series combination as a single impedance, we can draw the complex version of the circuit. Then it is familiar.

$$Z_1 = Z_{R1} + Z_C = R_1 + \frac{1}{j\omega C}$$



The complex expression for the gain of non-inverting amp would be:

$$G = \frac{\tilde{v}_o}{\tilde{v}_s} = 1 + \frac{Z_2}{Z_1}$$

$$G = 1 + \frac{Z_2}{Z_1} = 1 + \frac{R_2}{R_1 + \frac{1}{j\omega C}}$$ Proceed with caution.

$$= \frac{R_1 + \frac{1}{j\omega C}}{R_1 + \frac{1}{j\omega C}} + \frac{R_2}{R_1 + \frac{1}{j\omega C}}$$ It is easy to jump to wrong conclusions.

$$= \frac{R_1 + R_2 - j\left(\frac{1}{\omega C}\right)}{R_1 - j\left(\frac{1}{\omega C}\right)}$$

$$= \frac{23000\,\Omega - j1000\,\Omega}{1000\,\Omega - j1000\,\Omega}$$ Insert values

$$= \frac{(23002\,\Omega)\,e^{-j2.5^\circ}}{(1414\,\Omega)\,e^{-j45^\circ}} = 16.27e^{-j42.5^\circ}$$ Transform to polar and finish.

Then the complex output voltage is

$$\tilde{v}_o = G \cdot \tilde{v}_S = \left[(16.3)\,e^{j42.4^\circ}\right](0.5\,\mathrm{V}) = (8.13\,\mathrm{V})\,e^{j42.5^\circ}$$

This is an example where an extra math step leads to an instructive result. Starting at the step where

$$G = \frac{R_1 + R_2 - j\left(\frac{1}{\omega C}\right)}{R_1 - j\left(\frac{1}{\omega C}\right)}$$

$$= \frac{\frac{R_1 + R_2}{R_1} - j\left(\frac{1}{\omega R_1 C}\right)}{1 - j\left(\frac{1}{\omega R_1 C}\right)}$$

Divide top and bottom by R_1.

$$= \frac{\left(1 + \frac{R_2}{R_1}\right) - j\left(\frac{1}{\omega R_1 C}\right)}{1 - j\left(\frac{1}{\omega R_1 C}\right)}$$

Note how $1 + R_2/R_1$ appears.

$$= \frac{23 - j1}{1 - j1} = \frac{(23.02)\, e^{-j2.5^\circ}}{(1.414)\, e^{-j45^\circ}} = \left(16.3\right) e^{j42.5^\circ}$$